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AN

ESSAY

ON

LOGARITHMS:

DEDUCING

The whole DOCTRINE concerning Them

FROM

Pure ARITHMETICAL PRINCIPLES:

In a LETTER to JOHN GRAY, Eſq;

Rector of the MARISCHALL COLLEGE, NEW ABERDEEN, and F. R. S.

BY

ANDREW REID, Eſq; K

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TO

JOHN GRAY, Eſq;

Rector of the Mariſchall College, New Aberdeen, and F. R. S.

Dear Sir,

WHATEVER be the Merit of the *Eſſay* on *Logarithms*, which I herewith preſent you, it muſt be allowed to have ſome Claim to your Favour, as it was long ago compoſed, and is now publiſhed, chiefly at your Deſire. I hope, therefore, you will accept it as a Token of my Deference to your Judgment. I wiſh I durſt alſo flatter myſelf with the Hopes of its proving a laſting Memorial of that Friendſhip, which hath happily ſubſiſted between us, during a long Courſe of Years, and of that cordial Affection and perfect Eſteem, with which I am,

Dear Sir,

Your moſt faithful,

And moſt obedient humble Servant,

June 1ſt, 1767. ANDREW REID.

ADVERTISEMENT.

AFTER what the great *Halley* hath demonſtrated, in the *Philoſophical Tranſactions*, Nº 216, an Attempt to deduce the Doctrine of Logarithms from pure Arithmetical Principles, in a Way different from his, might appear quite ſuperfluous, and perhaps impertinent, had not ſome ingenious men declared publickly, that “ thoſe, who “ pretend to deduce Logarithms from the Properties of Numbers, “ are obliged to make uſe of ſo far-fetched Principles as they could “ never underſtand.” This reflexion ſeeming to be particularly aimed at Dr. *Halley's Compendious and Facile Method for Conſtructing the Logarithms*, and the Uſe he therein makes of *Ratios*, *Ratiunculæ*, *Fluxions*, and the *Roots of the infinite Powers*, I propoſed, to ſome Mathematical Gentlemen of my Acquaintance, another Method for obtaining the ſame End, without being liable to the ſame Objections. They approved of the Thought, and urged me to put it in Execution. I accordingly did ſo, near Twenty Years ago; but without any Deſign of publiſhing my Performance, till I was lately preſſed thereto, by ſome Friends, who were unwilling the Thing ſhould be altogether loſt. And this is the only Apology I can make for troubling the World with it.

I take this Opportunity of doing a Piece of Juſtice to the Memory of the illuſtrious *John Napier*, Baron of *Merchiſton*. Though he enjoys, without a Rival, the ſole Honour of having been the firſt Inventor of Logarithms; yet ſundry eminent Writers have, by an unaccountable Miſtake, robbed him of no inconſiderable Part of his juſt Praiſe, by pretending that Mr. *Henry Briggs*, then Profeſſor of Geometry in *Greſham* College, and afterwards *Savilian* Profeſſor of Geometry at *Oxford*, was the firſt who thought of the preſent Syſtem, which accordingly goes by his Name. Now is it not ſurpriſing, that an Error ſo injurious to Baron *Napier* ſhould have prevailed ſo generally, and ſo long, as it has done, directly contrary to Mr. *Briggs's* expreſs Teſtimony? For, in the Preface to his *Arithmetica Logarithmica* he writes thus:

“ Quod Logarithmi iſti diverſi ſunt ab iis, quos clariſſimus vir “ Baro *Merchiſtonii* in ſuo edidit *Canone mirifico*, non eſt quod mireris. “ Ego enim cum meis auditoribus *Londini*, publicè in Collegio “ *Greſhamenſi*, horum doctrinam explicarem; animadverti multo fu- “ turum commodius, ſi Logarithmus Sinus totius ſervaretur 0 (ut in “ *Canone mirifico*) Logarithmus autem, partis decimæ ejuſdem Sinus “ totius, nempe Sinus 5 graduum, 44′. 21″ eſſet 10000000000.

"Atque ea de re ſcripſi ſtatim ad ipſum Authorem, et quamprimum per anni tempus, et vacationem a publico docendi munere licuit, profectus ſum *Edinburgum*; ubi, humaniſſime ab eo acceptus, hæſi per integrum menſem. Cum autem inter nos de horum mutatione ſermo haberetur, ille ſe idem dudum ſenſiſſe, et cupiviſſe dicebat; veruntamen iſtos, quos jam paraverat, edendos curaſſe, donec alios, ſi per negotia et valetudinem liceret, magis commodos confeciſſet. Iſtam autem mutationem ita faciendam cenſebat, ut 0 eſſet Logarithmus Unitatis, et 10000000000 Sinus totius: quod ego longe commodiſſimum eſſe non potui non agnoſcere.

"Cœpi igitur, ejus hortatu, rejectis illis quos antea paraveram, de horum calculo ſerio cogitare: et ſequenti eſtate iterum profectus *Edinburgum*, horum quos hic exhibeo præcipuos illi oſtendi; idem etiam tertia æſtate libentiſſimè facturus, ſi Deus illum nobis tamdiu ſuperſtitem eſſe voluiſſet."

From hence it is manifeſt that Mr. *Briggs* did indeed propoſe a new Syſtem of Logarithms, but widely different from that now in Uſe; which was contrived ſolely by the Baron of *Merchiſtoun*, even before he publiſhed his *Canon Mirificus*, and by him communicated *viva voce* to Mr. *Briggs*, who, at the Baron's Requeſt, chearfully and ſucceſsfully undertook the Labour of the Calculation.

Some *Engliſh* Writers, miſtaking the Import of the Term Baron, have called this celebrated Perſon Lord *Napier*, a *Scotch* Nobleman. He was not indeed a Peer of *Scotland*; but the *Peerage of Scotland* informs us that he was of a very antient, honourable, and illuſtrious Family; that his Anceſtors, for many Generations, had been poſſeſſed of ſundry Baronies, and, amongſt others, of the Barony of *Merchiſtoun*, which deſcended to him by the Death of his Father in 1608. Mr. *Briggs*, therefore, very properly ſtiles him *Baro Merchiſtonii*. Now, according to *Skene*, *de Verborum ſignificatione*, "In this Realm (of *Scotland*) he is called an Barrone, quha haldis his Landes immediatelie in Chiefe of the King, and hes Power of Pit and Gallows; *Foſſa et Furca*; quhilk was firſt inſtitute and granted be King *Malcolme*, quha gave Power to the Barrones to have ane Pit, quhairin wemen condemned for thieft ſuld be drowned, and ane Gallows, quhairupon men thieves, and treſpaſſoures, ſuld be hanged, conforme to the Doome given in the Barron Court thereanent." So that a *Scotch* Baron, though no Peer, was nevertheleſs a very conſiderable Perſonage, both in Dignity and Power.

AN

ESSAY

ON

LOGARITHMS.

1. THAT the Doctrine of Logarithms depends wholly on Arithmetical Principles, and is moſt naturally to be deduced from them, I hope to evince in the following Sheets.

SECTION I.

Of the Method and Uſes of Notation by Indices.

2. IN the common Algebraic Notation, the Product of a multiplied into b is ab; the Product of ab into c is abc; and the Product of abc into d is $abcd$; and ſo on. In like Manner, the Product of z into z is zz; the Product of zz into z is zzz; and the Product of zzz into z is $zzzz$; and ſo on.

3. But when all the Factors are equal, it is much more commodious to write but one of them, with an Index affixed denoting their Number: Thus the Series, $z : zz : zzz : zzzz : zzzzz : zzzzzz$, &c. is much better expreſſed by $z^1 : z^2 : z^3 : z^4 : z^5 : z^6$, &c. ſo that if the Number of Factors be n, their Product will be z^n.

4. In ſuch a Series z^1 is called the Root; z^2 is called the 2d Power (or Square) of that Root; z^3 is its 3d Power (or Cube); z^4 its 4th Power; and z^n its nth Power.

5. Hence it is evident, that when two or more Powers of z are multiplied together, the Product is a new Power of z, having for its Index the Sum of the Indices of all the Factors. For in the ſame Manner as $zz \times zzz = zzzzz$; that is, as $z^2 \times z^3 = z^5 = z^{\overline{2+3}}$, ſo is $z^m \times z^n = z^{\overline{m+n}}$. So likewiſe is $z^m \times z^m = z^{2m}$, and $z^{2m} \times z^m = z^{3m}$ and $z^{3m} \times z^m = z^{4m}$, and $z^m \times z^m \times z^m$, &c. till the Number of Factors be n, is z^{nm}; where, as before, z^m is the Root; z^{2m} its Square, or 2d Power; z^{3m} its Cube, or 3d Power, and z^{nm} its nth Power.

6. Hence alſo it follows that, when any Power of z is divided by another Power of z, the Quotient will be a new Power of z, having for its Index what remains after ſubducting the Index of the Diviſor from the Index of the Dividend. For ſeeing $z^2 \times z^3 = z^5$, it is plain that $\frac{z^5}{z^2} = \frac{z^2 \times z^3}{z^2} = z^3 = z^{\overline{5-2}}$; and ſo is $\frac{z^m}{z^n} = z^{\overline{m-n}}$. Or, becauſe $m = n + \overline{m-n}$, therefore $z^m = z^{n+\overline{m-n}} = z^n \times z^{\overline{m-n}}$; and therefore $\frac{z^m}{z^n} = \frac{z^n \times z^{\overline{m-n}}}{z^n} = z^{\overline{m-n}}$.

7. From § 5, it appears that z^1, or any of its Powers, as z^m, may be conſidered as a Root, and raiſed to any aſſigned Power of ſuch Root, ſuppoſe its n^{th} Power, only by multiplying 1, or m, the Index of the given Root, into n the Index of the Power required. Thus the n^{th} Power of z^1 is z^n; and the n^{th} Power of z^m is z^{nm}.

8. And hence may any Root of any Power of z be extracted with eaſe. For, if the Square Root of z^n be deſired, then is z^n conſidered as a Square, and conſequently its Index n muſt be double the Index of its Root; ſo that $\sqrt[2]{z^n} = z^{\frac{n}{2}}$. If z^n be a Cube, its Index n is triple the Index of its Root; and $\sqrt[3]{z^n} = z^{\frac{n}{3}}$. If z^n be the m^{th} Power of any Root, then its Index n is m times the Index of that Root, and $\sqrt[m]{z^n} = z^{\frac{n}{m}}$.

9. By § 6, $\frac{z^1}{z^1} = z^{1-1} = z^0$: but $\frac{z^1}{z^1} = 1$; and therefore $z^0 = 1$. Whence $\frac{1}{z^1} = \frac{z^0}{z^1} = z^{0-1} = z^{-1}$; and $\frac{1}{z^2} = \frac{z^0}{z^2} = z^{-2}$, &c. So that the Geometrical Series $z^4 : z^3 : z^2 : z^1 : 1 : \frac{1}{z^1} : \frac{1}{z^2} : \frac{1}{z^3}$, &c. may be better expreſſed thus . . $z^4 : z^3 : z^2 : z^1 : z^0 : z^{-1} : z^{-2} : z^{-3}$, &c.

10. Since z^0 is always Unity, z^1 muſt be either greater or leſs than Unity. Let z^1 be put ſucceſſively equal to the Number 2, and to its reciprocal $\frac{1}{2}$; then will the Series $z^4 : z^3 : z^2 : z^1 : z^0 : z^{-1} : z^{-2} : z^{-3} : z^{-4}$, &c. repreſent, in the former Caſe, the Numbers $16 : 8 : 4 : 2 : 1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16}$, &c. and in the latter Caſe, their Reciprocals $\frac{1}{16} : \frac{1}{8} : \frac{1}{4} : \frac{1}{2} : 1 : 2 : 4 : 8 : 16$, &c. Where it is plain that, if z^1 be greater than Unity, then every Power of z, whoſe Index is greater than 0, will repreſent a Number greater than Unity; and alſo that every Power of z, whoſe Index is leſs than 0, will ſignify a proper Fraction, leſs than Unity: The contrary of all which will be true, when z^1 is taken leſs than Unity; for then the

greater the Affirmative Index of any Power of z is, the leſs will be the Number repreſented by it.

11. For our Purpoſe, therefore, Uniformity and Propriety of Expreſſion ſeem to require that z^1 ſhould be greater than Unity.

12. Having once fixed z to ſignify a particular Number, it is plain that the Powers of z may be employed with great advantage in Computations, inſtead of the Powers of that Number; and that a Sett of Tables, expreſſing the natural Numbers with the correſponding Powers of z, would greatly facilitate and ſhorten ſeveral of the moſt troubleſome and moſt tedious Operations in Arithmetic; Multiplication and Diviſion being performable by the bare Addition and Subtraction of Indices; while Powers are raiſed, and Roots extracted, by the eaſy Multiplication and Diviſion of Indices.

13. But it is alſo plain that, whatever Number you chuſe to make z^1 repreſent, ſuppoſe the Number A, it will require ſome Art and Pains to find the fractional Powers of z, correſponding to the Numbers that lie between 1 and A; A and A^2; A^2 and A^3, &c. To ſhew how that may be done, upon plain Arithmetical Principles, is the Deſign of this Eſſay.

SECTION II.

Of the Expanſion of Numbers into Series; particularly the Inveſtigation of the Binomial Theorem; with other Preliminaries.

TO prevent Interruption in our Reaſoning, it will not, perhaps, be amiſs if we here teach the young Student how to perform certain Operations that may be afterwards required of him.

To expand any given Number, $1+q$, *into an Infinite Series.*

14. Since $\frac{1}{1+q} = 1 - \frac{q}{1+q}$, if $\frac{q}{1+q}$ be called n, we have $\frac{1}{1+q} = 1 - n$, and $1 + q = \frac{1}{1-n} = 1 + n + n^2 + n^3 + n^4$, &c. *ad infinitum.*

15. In the ſame Manner, ſince $1 + q = \frac{1}{1-n}$, and ſo $1 + 2q + q^2 = \frac{1}{1-2n+n^2}$, if $\overline{2n - n^2}$ be called p, we have $1 + 2q + q^2 = \frac{1}{1-p} = 1 + p + p^2 + p^3 + p^4$, &c. *ad infinitum.*

16. Alſo becauſe $\frac{1}{1+n} = 1 - \frac{n}{1+n}$; if $\frac{n}{1+n}$ be called m, we ſhall have $\frac{1}{1+n} = 1 - m$, and $1 + n = \frac{1}{1-m} = 1 + m + m^2 + m^3 + m^4$, *&c. ad infinitum.*

17. Likewiſe ſince $\overline{1+q} = \frac{1}{1-n}$, and $\overline{1+n} = \frac{1}{1-m}$, we have $\overline{1+q} \times \overline{1+n} = \frac{1}{1-n} \times \frac{1}{1-m} = \frac{1}{1-m-n+mn}$. Let now $\overline{m+n}$ be called S, and for mn put P; then will $\overline{1+q} \times \overline{1+n} = \frac{1}{1-S+P}$. Again, let $S - P$ be called R; and then $\overline{1+q} \times \overline{1+n} = \frac{1}{1-R} = 1 + R + R^2 + R^3 + R^4$, *&c. ad infinitum.*

18. If the Equation $\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4$, *&c. ad infinitum*, be multiplied into $1 - z^n$, we ſhall have $\frac{1-z^n}{1-n} = 1 + z + z^2 + z^3 + z^4$, *&c.* to n Terms only; z^n and all the ſubſequent Terms vaniſhing.

19. In like Manner, if $\frac{1}{1-z^{\frac{n}{m}}} = 1 + z^{\frac{n}{m}} + z^{\frac{2n}{m}} + z^{\frac{3n}{m}} + z^{\frac{4n}{m}}$, *&c. ad infinitum*, be multiplied into $1 - z^{\frac{mn}{m}} = 1 - z^n$, we ſhall have $\frac{1-z^{\frac{mn}{m}}}{1-z^{\frac{n}{m}}} = 1 + z^{\frac{n}{m}} + z^{\frac{2n}{m}} + z^{\frac{3n}{m}} + z^{\frac{4n}{m}}$, *&c.* to m Terms only; $z^{\frac{mn}{m}}$ and all the ſubſequent Terms vaniſhing.

20. Hence the Fraction $\frac{1-z^{\frac{n}{m}}}{1-z}$, which is the Quotient of $\frac{1}{1-z}$ divided by $\frac{1}{1-z^{\frac{n}{m}}}$, that is, (multiplying both Diviſor and Dividend into $1-z^n = 1 - z^{\frac{mn}{m}}$) the Quotient of $\frac{1-z^n}{1-z}$ divided by $\frac{1-z^{\frac{mn}{m}}}{1-z^{\frac{n}{m}}}$, is =

$$\frac{1 + z + z^2 + z^3 + z^4, \text{ \&c. to } n \text{ Terms by § 18.}}{1 + z^{\frac{n}{m}} + z^{\frac{2n}{m}} + z^{\frac{3n}{m}}, \text{ \&c. to } m \text{ Terms by § 19.}}$$

21. And hence the Fraction $\frac{x^{\frac{n}{m}} - y^{\frac{n}{m}}}{x-y} = \frac{x^{\frac{n}{m}}}{x} \times \frac{1 - \frac{y^{\frac{n}{m}}}{x^{\frac{n}{m}}}}{1 - \frac{y}{x}}$, if $\frac{y}{x}$ be called z,

becomes $\frac{x^{\frac{n}{m}}}{x} \times \frac{1-z^{\frac{n}{m}}}{1-z} =$ (as above) $x^{\frac{n}{m}-1} \times \frac{1+z+z^2+z^3, \text{ \&c. to } n \text{ Terms}}{1+z^{\frac{n}{m}}+z^{\frac{2n}{m}}, \text{ \&c. to } m \text{ Terms.}}$

22. So alſo is $\frac{x^n-y^n}{x-y}$ (where m is 1) $= x^{n-1} \times 1+\frac{y}{x}+\frac{y^2}{x^2}+\frac{y^3}{x^3}$, &c. to n Terms: and if $x=y$, and ſo $\frac{y}{x}=1$, then is $x^{n-1} \times \overline{1+1+1}$, &c. to n Places $= x^{n-1} \times n = nx^{n-1}$.

To expand into an infinite Series the Fraction $\frac{q}{1+x}$, where q is given, and x unknown.

23. Firſt expand q, as in § 14, into the Series $n+n^2+n^3$, &c. and then divide this Series by $1+x$; obſerving only that every leſs Number may be expreſſed as a Part or Parts of a greater Number: Thus n^2-nx, being leſs than n^2, may be expreſſed by $\frac{n^2}{a}$, ſignifying either a Part, or ſome Parts, of n^2, according as a turns out to be an Integer or a Fraction. The Operation is as follows:

$$1+x \,\big|\, n+n^2+n^3+n^4+n^5+n^6 \text{ \&c.} \,\big|\, n+\frac{n^2}{a}+\frac{n^3}{b}+\frac{n^4}{c}+\frac{n^5}{d}+\frac{n^6}{e}, \text{ \&c.} = \frac{q}{1+x}$$

$$\begin{array}{l}
n+nx \\ \hline
n^2-nx=\frac{n^2}{a} \\
\frac{n^2}{a}+\frac{n^2x}{a} \\ \hline
n^3-\frac{n^2x}{a}=\frac{n^3}{b} \\
\frac{n^3}{b}+\frac{n^3x}{b} \\ \hline
n^4-\frac{n^3x}{b}=\frac{n^4}{c} \\
\frac{n^4}{c}+\frac{n^4x}{c} \\ \hline
n^5-\frac{n^4x}{c}=\frac{n^5}{d} \\
\frac{n^5}{d}+\frac{n^5x}{d} \\ \hline
n^6-\frac{n^5x}{d}=\frac{n^6}{e}, \text{ \&c.}
\end{array}$$

24. Or you may find the Quotient of $\frac{q}{1+x}$ thus: Whatever be the Value of x, it is plain that $1+x$ being greater than Unity, $\frac{q}{1+x}$

muſt be leſs than q, and conſequently ſome Part or Parts thereof; that is, ſome Part or Parts of each Member of the infinite Series $n+n^2+n^3+n^4+n^5+n^6$, *&c.* and, therefore, may be fitly expreſſed by $\frac{n}{a}+\frac{n^2}{b}+\frac{n^3}{c}+\frac{n^4}{d}+\frac{n^5}{e}+\frac{n^6}{f}$, *&c. ad infinitum.* And by either of theſe two Ways may any ſuch Fraction be expanded into an infinite Series.

To expand any Power of z into a Series.

25. Put $z=\overline{1+x}$ or $\overline{1-x}$, according as it is greater or leſs than Unity, and raiſe that Binomial, by continual Multiplication, to the Power aſſigned. Thus, putting $z=\overline{1+x}$, we have

$$\begin{aligned}
z^0 &= \overline{1+x}^0 = 1\\
z^1 &= \overline{1+x}^1 = 1+x\\
z^2 &= \overline{1+x}^2 = 1+2x+x^2\\
z^3 &= \overline{1+x}^3 = 1+3x+3x^2+x^3\\
z^4 &= \overline{1+x}^4 = 1+4x+6x^2+4x^3+x^4\\
z^5 &= \overline{1+x}^5 = 1+5x+10x^2+10x^3+5x^4+x^5\\
z^6 &= \overline{1+x}^6 = 1+6x+15x^2+20x^3+15x^4+6x^5+x^6\\
\&c.\quad & \&c.\qquad \&c.
\end{aligned}$$

26. This indeed would be a very tedious and troubleſome Way of coming at any deſired Power of $\overline{1+x}$, eſpecially a high one. But from theſe few Steps we may obſerve

27. Firſt, That the Powers of $\overline{1+x}$ being ſucceſſively produced by the continual Multiplication of the Root $\overline{1+x}$ into itſelf, the firſt Term of every Power will always be Unity.

28. Secondly, That any Power of $\overline{1+x}$ may be repreſented by this general Expreſſion, *viz.* $\overline{1+x}^a = 1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6$, *&c.* which being multiplied into the Root $1+x$ produces the next higher Power, *viz.*

$$\overline{1+x}^{a+1} = \left\{\begin{array}{l} 1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6, \ \&c. \\ \quad 1x+Ax^2+Bx^3+Cx^4+Dx^5+Ex^6, \ \&c. \end{array}\right\}$$ Whence it follows,

29. Thirdly, That the Sum of the *Unciæ* or Coefficients of the n^{th} and $\overline{n+1}^{th}$ Terms of the Series for $\overline{1+x}^a$ (whatever n or a be) is the *Uncia* or Coefficient of the $\overline{n+1}^{th}$ Term of the Series for $\overline{1+x}^{a+1}$, the next higher Power.

30. In order to try what Uſe can be made of theſe Hints, let the Letters, a, b, c, d, e, f, *&c.* in alphabetical Order, repreſent the natural

Numbers 1, 2, 3, 4, 5, 6, &c. as they riſe; then will $x=\frac{ax}{1}=\frac{bx}{2}=\frac{cx}{3}=\frac{dx}{4}=\frac{ex}{5}=\frac{fx}{6}$, &c, and the Geometrical Series $1+x+x^2+x^3+x^4+x^5+x^6$, &c. that is, $1+1x+1^2x^2+1^3x^3+1^4x^4+1^5x^5+1^6x^6$, &c. may be expreſſed thus, putting a Dot for the Mark of Multiplication: $1+\frac{a}{1}x+\frac{b.a}{1.2}x^2+\frac{c.b.a}{1.2.3}x^3+\frac{d.c.b.a}{1.2.3.4}x^4+\frac{e.d.c.b.a}{1.2.3.4.5}x^5+\frac{f.e.d.c.b.a}{1.2.3.4.5.6}x^6$, &c. by which Means every Power of x will have a Coefficient of the ſame Dimenſions with itſelf.

31. Now ſince, by § 27, Unity is the firſt Term of every Power of $\overline{1+x}$, write it down as ſuch, for as many Powers as you pleaſe; and then, by § 29, add together the Coefficients of the firſt and ſecond Terms of every Power, for the Coefficient of the ſecond Term of the next higher Power, and ſo on; thus making the Sum of the Coefficients of the n^{th} and $\overline{n+1}^{th}$ Terms of every Power the Coefficient of the $\overline{n+1}^{th}$ Term of the next higher Power, and obſerving, as by § 30, to write $\frac{a}{1}x$, $\frac{b.a}{1.2}x^2$, $\frac{c.b.a}{1.2.3}x^3$, $\frac{d.c.b.a}{1.2.3.4}x^4$, , &c. for x, x^2, x^3, x^4, &c. you will have the Table of Powers here ſubjoined, *viz.*

$$\overline{1+x}^{\,0}=1$$

$$\overline{1+x}^{\,a}=1+\frac{a}{1}x$$

$$\overline{1+x}^{\,b}=1+\frac{b}{1}x+\frac{b.a}{1.2}x^2$$

$$\overline{1+x}^{\,c}=1+\frac{c}{1}x+\frac{c.b}{1.2}x^2+\frac{c.b.a}{1.2.3}x^3$$

$$\overline{1+x}^{\,d}=1+\frac{d}{1}x+\frac{d.c}{1.2}x^2+\frac{d.c.b}{1.2.3}x^3+\frac{d.c.b.a}{1.2.3.4}x^4$$

$$\overline{1+x}^{\,e}=1+\frac{e}{1}x+\frac{e.d}{1.2}x^2+\frac{e.d.c}{1.2.3}x^3+\frac{e.d.c.b}{1.2.3.4}x^4+\frac{e.d.c.b.a}{1.2.3.4.5}x^5$$

$$\overline{1+x}^{\,f}=1+\frac{f}{1}x+\frac{f.e}{1.2}x^2+\frac{f.e.d}{1.2.3}x^3+\frac{f.e.d.c}{1.2.3.4}x^4+\frac{f.e.d.c.b}{1.2.3.4.5}x^5+\frac{f.e.d.c.b.a}{1.2.3.4.5.6}x^6, \text{ \&c.}$$

For $1+0=\frac{a}{1}$

$$1+\frac{a}{1}=\frac{b}{1}$$

$$1+\frac{b}{1}=\frac{c}{1}$$

$$1+\frac{c}{1}=\frac{d}{1}$$

$$1+\frac{d}{1}=\frac{e}{1}$$

$$1+\frac{e}{1}=\frac{f}{1}$$

Alſo $\frac{a}{1}+0=\frac{2.a}{1.2}=\frac{b.a}{1.2}$

$$\frac{b}{1}+\frac{b.a}{1.2}=\frac{2.b}{1.2}+\frac{a.b}{1.2}=\frac{c.b}{1.2}$$

$$\frac{c}{1}+\frac{c.b}{1.2}=\frac{2.c}{1.2}+\frac{b.c}{1.2}=\frac{d.c}{1.2}$$

$$\frac{d}{1}+\frac{d.c}{1.2}=\frac{2.d}{1.2}+\frac{c.d}{1.2}=\frac{e.d}{1.2}$$

$$\frac{e}{1}+\frac{e.d}{1.2}=\frac{2.e}{1.2}+\frac{d.e}{1.2}=\frac{f.d}{1.2}$$

Again $\frac{b.a}{1.2}+0=\frac{3.b.a}{1.2.3}=\frac{c.b.a}{1.2.3}$

$$\frac{c.b}{1.2}+\frac{c.b.a}{1.2.3}=\frac{3.c.b}{1.2.3}+\frac{a.c.b}{1.2.3}=\frac{d.c.b}{1.2.3}$$

$$\frac{d.c}{1.2}+\frac{d.c.b}{1.2.3}=\frac{3.d.c}{1.2.3}+\frac{b.d.c}{1.2.3}=\frac{e.d.c}{1.2.3}$$

$$\frac{e.d}{1.2}+\frac{e.d.c}{1.2.3}=\frac{3.e.d}{1.2.3}+\frac{c.e.d}{1.2.3}=\frac{f.e.d}{1.2.3}$$

&c.

Again $\frac{c.b.a}{1.2.3}+0=\frac{4.c.b.a}{1.2.3.4}=\frac{d.c.b.a}{1.2.3.4}$

$$\frac{d.c.b}{1.2.3}+\frac{d.c.b.a}{1.2.3.4}=\frac{4.d.c.b}{1.2.3.4}+\frac{a.d.c.b}{1.2.3.4}=\frac{e.d.c.b}{1.2.3.4}$$

$$\frac{e.d.c}{1.2.3}+\frac{e.d.c.b}{1.2.3.4}=\frac{4.e.d.c}{1.2.3.4}+\frac{b.e.d.c}{1.2.3.4}=\frac{f.e.d.c}{1.2.3.4}.$$

&c.

Again $\frac{d.c.b.a}{1.2.3.4}+0=\frac{5.d.c.b.a}{1.2.3.4.5}=\frac{e.d.c.b.a}{1.2.3.4.5}$

$$\frac{e.d.c.b}{1.2.3.4}+\frac{e.d.c.b.a}{1.2.3.4.5}=\frac{5.e.d.c.b}{1.2.3.4.5}+\frac{a.e.d.c.b}{1.2.3.4.5}=\frac{f.e.d.c.b}{1.2.3.4.5}$$

&c.

Again $\frac{e.d.c.b.a}{1.2.3.4.5}+0=\frac{6.e.d.c.b.a}{1.2.3.4.5.6}=\frac{f.e.d.c.b.a}{1.2.3.4.5.6}$

&c.

32. From theſe few Steps, the Law of Continuation is obvious enough; the Coefficients of every Term having a conſtant Relation to the Index of the Power of which it is a Member; ſo that any Power, $\overline{1+x}^{n}$, for Example, may be thus expreſſed $\overline{1+x}^{n}=1+\frac{n}{1}x+\frac{n.\overline{n-1}}{1.2}x^{2}+\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}x^{3}+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}x^{4}$, &c.

33. But to demonſtrate this important Rule more generally, let *a, b, c, d, &c.* be the Terms of an Arithmetical Progreſſion encreaſing by the common Difference z; and then will $\frac{a}{z}$, $\frac{b}{z}$, $\frac{c}{z}$, $\frac{d}{z}$, *&c.* be a Series of Arithmetical Progreſſionals, encreaſing continually by 1.

34. Now if each of theſe Terms be ſucceſſively divided by the natural Numbers 1, 2, 3, 4, *&c.* and 1 ſubtracted from each Quotient, we ſhall have the following Series, *viz.*

1. $\frac{a}{z}-1$; $\frac{a}{2z}-1$; $\frac{a}{3z}-1$; $\frac{a}{4z}-1$, *&c.* Or $\frac{a-z}{z}$; $\frac{a-2z}{2z}$; $\frac{a-3z}{3z}$; $\frac{a-4z}{4z}$, *&c.*
2. $\frac{b}{z}-1$; $\frac{b}{2z}-1$; $\frac{b}{3z}-1$; $\frac{b}{4z}-1$, *&c.* Or $\frac{b-z}{z}$; $\frac{b-2z}{2z}$; $\frac{b-3z}{3z}$; $\frac{b-4z}{4z}$, *&c.* And ſo of the reſt.

35. If the Terms of each of theſe Series be continually multiplied into one another, from Left to Right, it is plain that, in the firſt Series, putting M for the Product of m Terms, ($=\overline{n-1}$ Terms) the Product of n Terms will be $M\times\frac{a-nz}{nz}$; and ſo the Product of n Terms will be to the Product of m Terms as $M\times\frac{a-nz}{nz}$ to M; that is, as $\frac{a-nz}{nz}$ to 1; or as $a-nz$ to nz; and, therefore, the Product of n Terms will be to the Sum of the Products of n Terms and of m Terms as $a-nz$ to a.

36. But it is alſo plain that the ſame Product of n Terms, in the firſt Series, will be to the Product of n Terms, in the ſecond Series, as $a-nz$ to a: For, putting a Dot as the Mark of Multiplication, the

Product of n Terms in the firſt Series will be $\frac{a-z}{z}.\frac{a-2z}{2z}.\frac{a-3z}{3z}.\frac{a-4z}{4z}.$ &c. to $\frac{a-nz}{nz}$; and in the ſecond Series - - - $\frac{b-z}{z}.\frac{b-2z}{2z}.\frac{b-3z}{3z}.\frac{b-4z}{4z}.$ &c. to $\frac{b-nz}{nz}$: that is, the Denominators being the ſame in both, the former will be to the latter, as $\overline{a-z}.\ \overline{a-2z}.\ \overline{a-3z}.\ \overline{a-4z}.\ \overline{a-5z}.\ \overline{a-6z}.$ &c. to $\overline{a-nz}$ is to - $\overline{b-z}.\ \overline{b-2z}.\ \overline{b-3z}.\ \overline{b-4z}.\ \overline{b-5z}.\ \overline{b-6z}.$ &c. to $\overline{b-nz}$; which laſt is = to $a.\ \overline{a-z}.\ \overline{a-2z}.\ \overline{a-3z}.\ \overline{a-4z}.\ \overline{a-5z}.$ &c. to $\overline{a-mz}$: And therefore, dividing all by $a-z.\ a2z.\ a-3z.\ a-4z.\ a-5z.\ a-6z.$ &c. to $a-mz$, the Product of n Terms, in the firſt Series, will be to the Product of n Terms in the ſecond Series, as $a-nz$ to a.

37. Seeing therefore the Product of n Terms in the firſt Series hath the ſame Ratio to the Sum of the Products of n Terms and of m Terms in the ſaid firſt Series, that it hath to the Product of n Terms in the ſecond Series, it follows, that the Sum of the Products of n Terms and of m Terms in the firſt Series, is equal to the Product of n Terms in the ſecond.

38. The ſame will always hold true, if for $\frac{a}{z}$ and $\frac{b}{z}$ you ſubſtitute $\frac{b}{z}$ and $\frac{c}{z}$, $\frac{c}{z}$ and $\frac{d}{z}$, or any two contiguous Terms in the ſame Progreſſion. Nor will the Reaſoning or the Conſequence be different, if 1 be made the firſt Term of each Series, as it is a conſtant Factor in every Product, though it be needleſs to expreſs it as ſuch.

39. Making n therefore ſucceſſively equal to 1, 2, 3, 4, &c. the ſucceſſive Products of n Terms will ſtand thus:

$$1+\frac{a-z}{z}+\frac{a-z}{z}.\frac{a-2z}{2z}+\frac{a-z}{z}.\frac{a-2z}{2z}.\frac{a-3z}{3z}+\frac{a-z}{z}.\frac{a-2z}{2z}.\frac{a-3z}{3z}.\frac{a-4z}{4z}+\text{\&c.}$$

$$1+\frac{b-z}{z}+\frac{b-z}{z}.\frac{b-2z}{2z}+\frac{b-z}{z}.\frac{b-2z}{2z}.\frac{b-3z}{3z}+\frac{b-z}{z}.\frac{b-2z}{2z}.\frac{b-3z}{3z}.\frac{b-4z}{4z}+\text{\&c.}$$

40. Now if each of theſe Terms be multiplied into its correſponding Term in the Geometrical Series 1, x^1, x^2, x^3, x^4, &c. you will have

$$1+\frac{a-z}{z}x+\frac{a-z}{z}.\frac{a-2z}{2z}x^2+\frac{a-z}{z}.\frac{a-2z}{2z}.\frac{a-3z}{3z}x^3+\frac{a-z}{z}.\frac{a-2z}{2z}.\frac{a-3z}{3z}.\frac{a-4z}{4z}x^4\ \text{\&c.}$$

$$1+\frac{b-z}{z}x+\frac{b-z}{z}.\frac{b-2z}{2z}x^2+\frac{b-z}{z}.\frac{b-2z}{2z}.\frac{b-3z}{3z}x^3+\frac{b-z}{z}.\frac{b-2z}{2z}.\frac{b-3z}{3z}.\frac{b-4z}{4z}x^4\ \text{\&c.}$$

Where it is manifeſt, from what hath been proved above, that the Sum of the Coefficients of x^m and of x^n, in the former Series, is always equal to the Coefficient of x^n in the latter. Whence it follows, by § 28, and § 37, that if the former Series be multiplied into $\overline{1+x}$, the latter Series will be produced: for the Product will be

$$\begin{array}{l} \left\{\begin{array}{l} 1+\frac{a-z}{z}x+\frac{a-z}{z}\cdot\frac{a-2z}{2z}x^2+\frac{a-z}{z}\cdot\frac{a-2z}{2z}\cdot\frac{a-3z}{3z}x^3+\frac{a-z}{z}\cdot\frac{a-2z}{2z}\cdot\frac{a-3z}{3z}\cdot\frac{a-4z}{4z}x^4,\ \&c. \\ +1x+\frac{a-z}{z}x^2+\frac{a-z}{z}\cdot\frac{a-2z}{2z}\cdot x^3+\frac{a-z}{z}\cdot\frac{a-2z}{2z}\cdot\frac{a-3z}{3z}\cdot x^4,\ \&c. \end{array}\right\} \\ \hline =1+\frac{b-z}{z}x+\frac{b-z}{z}\cdot\frac{b-2z}{2z}x^2+\frac{b-z}{z}\cdot\frac{b-2z}{2z}\cdot\frac{b-3z}{3z}x^3+\frac{b-z}{z}\cdot\frac{b-2z}{2z}\cdot\frac{b-3z}{3z}\cdot\frac{b-4z}{4z}x^4,\ \&c. \end{array}$$

For $\frac{a-z}{z}+1=\frac{a}{z}=\frac{b-z}{z}$: And $\frac{a-2z}{2z}+1\times\frac{a-z}{z}=\frac{a}{2z}\times\frac{a-z}{z}=\frac{b-z}{z}\cdot\frac{b-2z}{2z}$: And $\frac{a-3z}{3z}+1\times\frac{a-z}{z}\cdot\frac{a-2z}{2z}=\frac{a}{3z}\times\frac{a-z}{z}\times\frac{a-2z}{2z}=\frac{b-z}{z}\cdot\frac{b-2z}{2z}\cdot\frac{b-3z}{3z}$, &c. And if this Product be again multiplied into $\overline{1+x}$, the Series ariſing will be $1+\frac{c-z}{z}x+\frac{c-z}{z}\cdot\frac{c-2z}{2z}x^2$, &c. and ſo on *ad infinitum*.

41. Thus far is certain, let the values of a and z be what you will, integral or fractional. Let us now ſuppoſe $a=z=1$: then will a, b, c, d, &c. repreſent the natural Numbers 1, 2, 3, 4, &c. and (putting $\alpha=0$) $\frac{a-z}{z}$; $\frac{b-z}{z}$; $\frac{c-z}{z}$; $\frac{d-z}{z}$, &c. will be reſpectively $\frac{\alpha}{1}, \frac{a}{1}, \frac{b}{1}, \frac{c}{1}$, &c. which being introduced in their ſtead, and 1 inſtead of z, we ſhall have, with no other change but that of the Notation,

$$1+\frac{\alpha}{1}x+\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}x^2+\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}x^3+\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}\cdot\frac{\alpha-3}{4}x^4,\ \&c.$$

$$1+\frac{a}{1}x+\frac{a}{1}\cdot\frac{a-1}{2}x^2+\frac{a}{1}\cdot\frac{a-1}{2}\cdot\frac{a-2}{3}x^3+\frac{a}{1}\cdot\frac{a-1}{2}\cdot\frac{a-2}{3}\cdot\frac{a-3}{4}x^4,\ \&c.$$

$$1+\frac{b}{1}x+\frac{b}{1}\cdot\frac{b-1}{2}x^2+\frac{b}{1}\cdot\frac{b-1}{2}\cdot\frac{b-2}{3}x^3+\frac{b}{1}\cdot\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot\frac{b-3}{4}x^4,\ \&c.$$ And ſo on.

42. But if $\frac{a-z}{z}=\frac{a-a}{a}=\frac{1-1}{1}=\frac{\alpha}{1}=0$, all the Terms in which $\frac{\alpha}{1}$ is a Factor will vaniſh, and then $\overline{1+x}^0=1$ will be $\overline{1+x}^\alpha=1+\frac{\alpha}{1}x+\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}x^2+\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}x^3+\frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}\cdot\frac{\alpha-3}{4}x^4$, &c. and both Sides of this Equation being multiplied continually into $1+x$, the ſucceſſive Products will be

$$\overline{1+x}^{a}=1+\frac{a}{1}x+\frac{a}{1}\cdot\frac{a-1}{2}x^2+\frac{a}{1}\cdot\frac{a-1}{2}\cdot\frac{a-2}{3}x^3+\frac{a}{1}\cdot\frac{a-1}{2}\cdot\frac{a-2}{3}\cdot\frac{a-3}{4}x^4,\ \&c.$$

$$\overline{1+x}^{b}=1+\frac{b}{1}x+\frac{b}{1}\cdot\frac{b-1}{2}x^2+\frac{b}{1}\cdot\frac{b-1}{2}\cdot\frac{b-2}{3}x^3+\frac{b}{1}\cdot\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot\frac{b-3}{4}x^4,\ \&c.$$

$$\overline{1+x}^{c}=1+\frac{c}{1}x+\frac{c}{1}\cdot\frac{c-1}{2}x^2+\frac{c}{1}\cdot\frac{c-1}{2}\cdot\frac{c-2}{3}x^3+\frac{c}{1}\cdot\frac{c-1}{2}\cdot\frac{c-2}{3}\cdot\frac{c-3}{4}x^4,\ \&c.$$

And generally, if n be any Term in the infinite Arithmetical Series $a, b, c, d,$ &c. we ſhall find

$\overline{1+x}^{n}=1+\frac{n}{1}x+\frac{n}{1}\cdot\frac{n-1}{2}x^2+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4$, &c. as before.

43. This being ſo, it is plain that, if n be changed into $\frac{n}{m}$ (for Example) on one Side of the Equation, it muſt neceſſarily be changed in like manner into $\frac{n}{m}$ on the other; ſo that

$$\overline{1+x}^{\frac{n}{m}}=1+\frac{\frac{n}{m}}{1}x+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}x^2+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}\cdot\frac{\frac{n}{m}-2}{3}x^3+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}\cdot\frac{\frac{n}{m}-2}{3}\cdot\frac{\frac{n}{m}-3}{4}x^4,\ \&c.$$

44. For the ſame reaſon, if n becomes $-n$ on one ſide, it muſt undergo the ſame change on the other: and ſo likewiſe muſt x have the ſame Sign on both ſides. Thus have we obtained a general Rule for expanding any given Power of x, that is of $\overline{1+x}$ or $\overline{1-x}$, into a Series, whether the Index of that Power be integral or fractional, poſitive, or negative: And this is the celebrated Binomial Theorem.

45. Since $\overline{y+x}=y\times\overline{1+\frac{x}{y}}$, therefore $\overline{y+x}^{n}=y^{n}\times\overline{1+\frac{x}{y}}^{n}$: that is,

$\overline{y+x}^{n}=$ to $y^{n}\times 1+\frac{nx}{1y}+\frac{n}{1}\cdot\frac{n-1}{2}\frac{x^2}{y^2}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\frac{x^3}{y^3}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\frac{x^4}{y^4}$, &c.

Or to $y^{n}+\frac{n}{1}xy^{\overline{n-1}}+\frac{n}{1}\cdot\frac{n-1}{2}x^2y^{\overline{n-2}}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3y^{\overline{n-3}}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4y^{\overline{n-4}}$, &c.

46. Seeing the Coefficients, or *Unciæ*, of the Powers of x in $\overline{1+x}^{n}$ are the ſame, whatever be the Value of x, let $x=1$, and then we ſhall have $\overline{1+x}^{n}=\overline{1+1}^{n}=2^{n}=1+\frac{n}{1}+\frac{n}{1}\cdot\frac{n-1}{2}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}$, &c. And it is eaſy to ſee that the Sum of all the Coefficients in $\overline{1+x+y}^{n}$ expanded into a Series is, in like manner, equal to 3^{n}; and ſo on of any Multinomial.

47. If for $n.\,n-1.\,n-2.\,n-3$, &c. be put $n.m.l.k$, &c. we ſhall have $\overline{1+x}^n = 1 + \frac{nx}{1} + \frac{nm}{1.2}x^2 + \frac{nml}{1.2.3}x^3 + \frac{nmlk}{1.2.3.4}x^4$, &c. Now ſuppoſe a Table of the integral Powers of $\overline{1+x}$ formed, as in § 31, by Addition, and the inſignificant Terms omitted, thus:

$$\begin{array}{l}
\overline{1+x}^0 = 1 \\
\overline{1+x}^a = 1 + \frac{ax}{1} \\
\overline{1+x}^b = 1 + \frac{bx}{1} + \frac{ab}{1.2}x^2 \\
\overline{1+x}^c = 1 + \frac{cx}{1} + \frac{bc}{1.2}x^2 + \frac{abc}{1.2.3}x^3 \\
\overline{1+x}^d = 1 + \frac{dx}{1} + \frac{cd}{1.2}x^2 + \frac{bcd}{1.2.3}x^3 + \frac{abcd}{1.2.3\,4}x^4 \\
\overline{1+x}^e = 1 + \frac{ex}{1} + \frac{de}{1.2}x^2 + \frac{cde}{1.2.3}x^3 + \frac{bcde}{1.2.3.4}x^4 + \frac{abcde}{1.2.3.4.5}x^5 \\
\overline{1+x}^f = 1 + \frac{fx}{1} + \frac{ef}{1.2}x^2 + \frac{def}{1.2.3}x^3 + \frac{cdef}{1.2.3.4}x^4 + \frac{bcdef}{1.2.3.4.5}x^5 + \frac{abcdef}{1.2.3.4.5.6}x^6 \\
\hline
\overline{1+x}^g = 1 + \frac{gx}{1} + \frac{fg}{1.2}x^2 + \frac{efg}{1.2.3}x^3 + \frac{defg}{1.2.3.4}x^4 + \frac{cdefg}{1.2.3.4.5}x^5 + \frac{bcdefg}{1.2.3.4.5.6}x^6 + \frac{abcdefg}{1.2.3.4.5.6.7}x
\end{array}$$

and having drawn a black line, between $\overline{1+x}^f$ and $\overline{1+x}^g$ (for Example) ſo as to ſeparate all the preceding Series from the next following, it will appear by inſpection that the perpendicular Columns conſiſt, ſucceſſively from Left to Right, of the Coefficients, or *Unciæ*, of x^0, x^1, x^2, x^3, x^4, &c. and that the Number of Terms in each Column, above the Line, is reſpectively g, f, e, d, c, b, &c. or the difference between the Index of x in that Column, and g the Index of $\overline{1+x}^g$.

48. From the Conſtruction of the Table by Addition it appears clearly,

Firſt, That the *Unciæ* in each Column are the Differences between the *Unciæ* in its right Hand Column.

Secondly, That 1 is therefore the 1ſt, 2d, 3d, 4th, &c. Difference between the *Unciæ* of x^1, x^2, x^3, x^4, &c. reſpectively.

Thirdly, That the Coefficient of x^m in any Series is the Sum of all the Coefficients of x^{m-1} preceeding that Series.

Fourthly, That conſequently $\frac{fg}{1.2}$ is the Sum of a Series of f Terms, beginning with 1, whoſe 1ſt Difference is 1: or, putting N for f the Number of Terms, the Sum is $\frac{N.N+1}{1.\ 2}$. Alſo that $\frac{efg}{1.2.3}$ is the Sum of a Series of e Terms, beginning with 1, whoſe 2d Difference is 1: or, putting N for the Number of Terms, it is $\frac{N.N+1.N+2}{1.\ 2.\ 3.}$. Again, $\frac{defg}{1.2.3.4}$ is the Sum of a Series of d Terms, beginning with 1, whoſe 3d Difference is 1: or, putting N for the Number of Terms, it is $\frac{N.N+1.N+2.N+3}{1.\ 2.\ 3.\ 4}$. And in general the Sum of a Series of N Terms, beginning with 1, whoſe n^{th} Difference is 1, is $\frac{N.N+1.N+2,\ \&c.\ \text{to}\ N+n}{1.\ 2.\ 3\ \&c.\ \text{to}\ n+1}$.

Fifthly, If the firſt Term, and the 1ſt, 2d, 3d, 4th, &c. Difference be A, the Sum may be had by multiplying A into the Sum found as above.

Sixthly, Seeing the Coefficients of x^2 are $\frac{ab}{1.2}+\frac{bc}{1.2}+\frac{cd}{1.2}$, &c. to $\frac{ef}{1.2}$, the Sum whereof (the Number of Terms being $e = N$) is $\frac{efg}{1.2.3} = \frac{N.N+1.N+2}{1.\ 2.\ 3.}$; it is plain that multiplying all into 2, the Sum of the Products $ab + bc + cd$, &c. to N Terms is $\frac{N.N+1.N+2}{3}$. Alſo, ſeeing the Coefficients of x^3 are $\frac{abc}{1.2.3}+\frac{bcd}{1.2.3}+\frac{cde}{1.2.3}$, &c. to $\frac{def}{1.2.3}$, the Number of Terms being $d = N$, and their Sum $\frac{defg}{1.2.3.4} = \frac{N.N+1.N+2.N+3}{1.\ 2.\ 3.\ 4}$, it is clear that, multiplying all into 2×3, the Sum of the Products $abc + bcd + cde$, &c. to N Terms will be $\frac{N.N+1.N+2.N+3}{4}$. And generally, if n be the Number of Factors ſucceſſively multiplied together, 1 being the 1ſt, and alſo the common Difference between them, and if N be the Number of Products, the Sum of thoſe Products will be $\frac{N.N+1.N+2,\ \&c.\ \text{to}\ N+n}{n+1}$.

49. Having ſhewn above, in § 17, that $\overline{1+q} \times \overline{1+n} = \frac{1}{1-n} \times \frac{1}{1-m}$ $= \frac{1}{1-n-m+nm} = \frac{1}{1-S+P} = \frac{1}{1-R} = 1+R+R^2+R^3+R^4+R^5+R^6$, &c. and as it may afterwards be required to reduce R, and its Powers, to the Powers of m and n, it ſeems beſt to ſhew here how thoſe reductions may be made, in the eaſieſt and conciſeſt manner, that is ſuitable to our purpoſe, to avoid the being incumbered therewith in a more inconvenient place. Firſt then,

Since $S=m+n$ and $P=mn$, by § 17, we have

$S^2=m^2+n^2+2P$
$S^3=m^3+n^3+3SP$
$S^4=m^4+n^4+4S^2P-2P^2$
$S^5=m^5+n^5+5S^3P-5SP^2$
$S^6=m^6+n^6+6S^4P-9S^2P^2+2P^3$
$S^7=m^7+n^7+7S^5P-14S^3P^2+7SP^3$
* $S^8=m^8+n^8+8S^6P-20S^4P^2+16S^2P^3$, &c.

Secondly, ſince by § 17 $R=\overline{S-P}$, its Powers are found by § 45, and the preceding to be

$R=S-P$
$R^2=S^2-2SP+P^2$
$R^3=S^3-3S^2P+3SP^2-P^3$
$R^4=S^4-4S^3P+6S^2P^2-4SP^3+P^4$
$R^5=S^5-5S^4P+10S^3P^2-10S^2P^3+5SP^4-P^5$
$R^6=S^6-6S^5P+15S^4P^2-20S^3P^3+15S^2P^4-6SP^5+P^6$
$R^7=S^7-7S^6P+21S^5P^2-35S^4P^3+35S^3P^4-21S^2P^5+7SP^6-P^7$
$R^8=S^8-8S^7P+28S^6P^2-56S^5P^3+70S^4P^4-56S^3P^5+28S^2P^6-8SP^7+P8$.
&c. &c.

* Theſe Reductions are eaſily made by multiplying the Powers of m and of n into $m+n$, and the remaining Terms into S; and expreſſing by the Powers of S and P all but the ſimple Powers of m and n. Thus for Example $\overline{m^3+n^3} \times \overline{m+n}$ produces $m^4+n^4+m^3n+n^3m$. Now $m^3n+n^3m = mn \times \overline{m^2+n^2} = P \times S^2-2P = S^2P-2P^2$; to which adding, $S \times 3SP = 3S^2P$, you will have $S^4 = m^4+n^4+4S^2P-2P^2$.

Thirdly,

ly, Now if for S, and the ſimple Powers of S, we inſert here their Values bove, we ſhall have the Reduction deſired. For ſince by § 17, n, and $R=S-P$, we have

P

$n^2+2P-2SP+P^2$

$n^3+3SP-3S^2P+3SP^2-P^3$

$n^4+4S^2P-2P^2-4S^3P+6S^2P^2-4SP^3+P^4$

$n^5+5S^3P-5SP^2-5S^4P+10S^3P^2-10S^2P^3+5SP^4-P^5$

$n^6+6S^4P-9S^2P^2+2P^3-6S^5P+15S^4P^2-20S^3P^3+15S^2P^4-6SP^5+P^6$

$n^7+7S^5P-14S^3P^2+7SP^3-7S^6P+21S^5P^2-35S^4P^3+35S^3P^4-21S^2P^5+7SP^6-P^7$

$n^8+8S^6P-20S^4P^2+16S^2P^3-8S^7P+28S^6P^2-56S^5P^3+70S^4P^4-56S^3P^5+28S^2P^6-8SP^7+P^8.$

SECTION

SECTION III.

Containing two neceſſary Lemmas; with Mr. Landen's *Algebraical Demonſtration of the Binomial Theorem.*

Lemma I.

50. *In the Equation* $\frac{n}{x}+\frac{n^2}{y}+\frac{n^3}{z}$, *&c.* $=\frac{n}{r}+\frac{n^2}{s}+\frac{n^3}{t}$, *&c. where n is invariable, the Coefficients of the ſame Powers of n are equal.* For ſince $\frac{n}{x}+\frac{n^2}{y}+\frac{n^3}{z}$, *&c.* $=\frac{n}{r}+\frac{n^2}{s}+\frac{n^3}{t}$, *&c.* therefore $\frac{n}{x}-\frac{n}{r}=\left.\begin{array}{l}\frac{n^2}{s}+\frac{n^3}{t},\ \&c.\\ -\frac{n^2}{y}-\frac{n^3}{z},\ \&c.\end{array}\right\}$ and $1:n::\frac{1}{x}-\frac{1}{r}:\left.\begin{array}{l}\frac{n^2}{s}+\frac{n^3}{t},\ \&c.\\ -\frac{n^2}{y}-\frac{n^3}{z},\ \&c.\end{array}\right\}$: Where $\frac{1}{x}-\frac{1}{r}$, the Difference between the Coefficients of *n*, being quite unlimited, is therefore arbitrary, and may be what you pleaſe, or nothing; that is, $\frac{1}{x}-\frac{1}{r}=0$ muſt be one of its innumerable Values, and then is $\frac{1}{x}=\frac{1}{r}$. Alſo $\left.\begin{array}{l}\frac{n^2}{s}+\frac{n^3}{t},\ \&c.\\ -\frac{n^2}{y}-\frac{n^3}{z},\ \&c.\end{array}\right\}=0$; or $\frac{n^2}{s}-\frac{n^2}{y}=\frac{n^3}{z}-\frac{n^3}{t}$, *&c.* Conſequently $1:n^2::\frac{1}{s}-\frac{1}{y}:\frac{n^3}{z}-\frac{n^3}{t}$, *&c.* where again $\frac{1}{s}-\frac{1}{y}$, the Difference betweeen the Coefficients of n^2, being quite unlimited, is therefore arbitrary, and $\frac{1}{s}-\frac{1}{y}=0$ is one of its innumerable Values, ſo that $\frac{1}{s}=\frac{1}{y}$. Alſo $\frac{n^3}{z}-\frac{n^3}{t}$ muſt be $=0$; and $\frac{1}{z}=\frac{1}{t}$; and ſo on.

51. Corol. The like reaſoning will prove that, in an Equation of this Form, $\left\{\begin{matrix}\frac{n}{x}+\frac{n^2}{y}+\frac{n^3}{z}, \text{\&c.}\\ +\frac{m}{\varphi}+\frac{m^2}{\chi}+\frac{m^3}{\psi}, \text{\&c.}\end{matrix}\right\}=\left\{\begin{matrix}\frac{n}{r}+\frac{n^2}{s}+\frac{n^3}{t}, \text{\&c.}\\ +\frac{m}{u}+\frac{m^2}{v}+\frac{m^3}{w}, \text{\&c.}\end{matrix}\right\}$ where there are two, or more, invariable Quantities on each Side, the Coefficients of the ſame Powers, of the ſame conſtant Quantities, are always equal.

LEMMA II.

52. In the Equation $\frac{a}{x}-\frac{b}{y}=c$ (where a, b, c are known Quantities) $x=y=\frac{a-b}{c}$. For ſince $\frac{a}{x}-\frac{b}{y}=c=\overline{a-b}\times\frac{c}{a-b}$, let $\frac{c}{a-b}=\frac{1}{r}$; and then $\frac{a}{x}-\frac{b}{y}=\frac{a}{r}-\frac{b}{r}$; and $\frac{a}{x}-\frac{a}{r}=\frac{b}{y}-\frac{b}{r}$; and $a : b :: \frac{1}{y}-\frac{1}{r} : \frac{1}{x}-\frac{1}{r}$: where $\frac{1}{y}-\frac{1}{r}$ being unlimited hath innumerable values, of which one is $\frac{1}{y}-\frac{1}{r}=0$; in which Caſe $\frac{1}{x}-\frac{1}{r}=0$ alſo: and ſo $y=x=r=\frac{a-b}{c}$.

53. Though the Binomial Theorem hath been fully demonſtrated above, yet, perhaps, it may not be ſuperfluous to add here another Demonſtration thereof, which the ingenious Mr. *Landen* hath given us in his *Reſidual Analyſis*; eſpecially as it is deduced entirely from Arithmetical Principles.

It appeared, § 28. that we may put any Power of $\overline{1+x}$, ſuppoſe

$\overline{1+x}^{\frac{m}{n}}=1+Ax+Bx^2+Cx^3+Dx^4$, &c. and therefore

$\overline{1+y}^{\frac{m}{n}}=\underline{1+Ay+By^2+Cy^3+Dy^4, \text{\&c.}}$ Hence by Subtraction

$\overline{1+x}^{\frac{m}{n}}-\overline{1+y}^{\frac{m}{n}}=A\times\overline{x-y}+B\times\overline{x^2-y^2}+C\times\overline{x^3-y^3}+D\times\overline{x^4-y^4}$, &c. and dividing all by

$\overline{1+x}-\overline{1+y}=x-y$ we ſhall have by § 22.

$$\frac{\overline{1+x}^{\frac{m}{n}}-\overline{1+y}^{\frac{m}{n}}}{\overline{1+x}-\overline{1+y}}=A+B\times\overline{x+y}+C\times\overline{x^2+xy+y^2}+D\times\overline{x^3+x^2y+xy^2+y^3}, \text{\&c.}$$

54. Now $\frac{\overline{1+x}^{\frac{n}{m}}-\overline{1+y}^{\frac{n}{m}}}{\overline{1+x}-\overline{1+y}}=\frac{\overline{1+x}^{\frac{n}{m}}}{\overline{1+x}}\times\frac{1-\frac{\overline{1+y}^{\frac{n}{m}}}{\overline{1+x}^{\frac{n}{m}}}}{1-\frac{1+y}{1+x}}$; and putting $\frac{1+y}{1+x}$ $=z$, and ſo $\frac{\overline{1+y}^{\frac{n}{m}}}{\overline{1+x}^{\frac{n}{m}}}=z^{\frac{n}{m}}$, we have $\frac{\overline{1+x}^{\frac{n}{m}}-\overline{1+y}^{\frac{n}{m}}}{\overline{1+x}-\overline{1+y}}=\frac{\overline{1+x}^{\frac{n}{m}}}{\overline{1+x}}\times\frac{1-z^{\frac{n}{m}}}{1-z}$

$$=\frac{\overline{1+x}^{\frac{n}{m}}}{1+x}\times\left.\frac{1+z+z^2+z^3,\ \&c.\ \text{to } n \text{ Places}}{1+z^{\frac{n}{m}}+z^{\frac{2n}{m}}+z^{\frac{3n}{m}},\ \&c.\ \text{to } m \text{ Places}}\right\}\text{by § 20.}$$

55. This is true, whatever be the values of x and y, and conſequently is ſo when they are equal. But in that caſe $\frac{1+y}{1+x}=z=1$; and every Power of z is alſo 1: ſo that

$$\left.\frac{1+z+z^2+z^3,\ \&c.\ \text{to } n \text{ Places}}{1+z^{\frac{n}{m}}+z^{\frac{2n}{m}}+z^{\frac{3n}{m}},\ \&c.\ \text{to } m \text{ Places}}\right\}=\frac{n}{m};\ \text{and}\ \frac{\overline{1+x}^{\frac{n}{m}}}{1+x}\times\left.\frac{1+z+z^2+z^3,\ \&c.\ \text{to } n \text{ Places}}{1+z^{\frac{n}{m}}+z^{\frac{2n}{m}}+z^{\frac{3n}{m}},\ \&c.\ \text{to } m \text{ Places}}\right\}$$

$$=\frac{\overline{1+x}^{\frac{n}{m}}}{1+x}\times\frac{n}{m}=\frac{\overline{1+x}^{\frac{n}{m}}-\overline{1+y}^{\frac{n}{m}}}{\overline{1+x}-\overline{1+y}}.$$

56. Therefore, ſince $x=y$, we have, by § 53, $\frac{\overline{1+x}^{\frac{n}{m}}}{1+x}\times\frac{n}{m}$ $=A+2Bx+3Cx^2+4Dx^3$, &c. and multiplying all by $1+x$, we have, $\overline{1+x}^{\frac{n}{m}}\times\frac{n}{m}=\left\{\begin{array}{l}A+2Bx+3Cx^2+4Dx^3,\ \&c.\\ \quad+Ax+2Bx^2+3Cx^3,\ \&c.\end{array}\right\}$.

57. But, by § 53. $\overline{1+x}^{\frac{n}{m}}=1+Ax+Bx^2+Cx^3+Dx^4$, &c. and therefore $\overline{1+x}^{\frac{n}{m}}\times\frac{n}{m}=\frac{n}{m}+\frac{n}{m}Ax+\frac{n}{m}Bx^2+\frac{n}{m}Cx^3+\frac{n}{m}Dx^4$, &c. and theſe two values of $\overline{1+x}^{\frac{n}{m}}\times\frac{n}{m}$ being equal, we have

$$\begin{array}{lllll}A & +2Bx & +3Cx^2 & +4Dx^3, \&c. \\ & +A & +2B & +3C\end{array}=\frac{n}{m}+\frac{n}{m}Ax+\frac{n}{m}Bx^2+\frac{n}{m}Cx^3,\ \&c.\ \text{And by § 50.}$$

$$A=\frac{n}{m}$$

$$2B+A=\frac{n}{m}A;\ \text{and}\ 2B=\frac{n}{m}A-A=A\times\frac{n}{m}-1=\frac{n}{m}\times\frac{n}{m}-1:\ \text{and}\ B=\frac{\frac{n}{m}\cdot\overline{\frac{n}{m}-1}}{1.\ 2}$$

$$3C+2B=\frac{n}{m}B;\ \text{and}\ 3C=\frac{n}{m}B-2B=B\times\frac{n}{m}-2=\frac{\frac{n}{m}\cdot\overline{\frac{n}{m}-1}}{1.\ 2}\cdot\overline{\frac{n}{m}-2}:\ \text{and}\ C=\frac{\frac{n}{m}\cdot\overline{\frac{n}{m}-1}\cdot\overline{\frac{n}{m}-2}}{1.\ 2.\ 3}$$

$$4D+3C=\frac{n}{m}C;\ \text{and}\ 4D=\frac{n}{m}C-3C=C\times\frac{n}{m}-3=\frac{\frac{n}{m}\cdot\overline{\frac{n}{m}-1}\cdot\overline{\frac{n}{m}-2}}{1.\ 2.\ 3}\cdot\overline{\frac{n}{m}-3}:\ \text{and}\ D=\frac{\frac{n}{m}\cdot\overline{\frac{n}{m}-1}\cdot\overline{\frac{n}{m}-2}\cdot\overline{\frac{n}{m}-3}}{1.\ 2.\ 3.\ 4},\ \&c.$$

58. As theſe few Steps clearly point out the Law of Continuation, it follows generally that, if $\frac{n}{m}$ be the Index of any Power of $\overline{1+x}$, whether it be integral or fractional, then ſhall

$$\overline{1+x}^{\frac{n}{m}} = 1 + \frac{n}{m}x + \frac{\frac{n}{m}.\overline{\frac{n}{m}-1}}{1.\ 2}x^2 + \frac{\frac{n}{m}.\overline{\frac{n}{m}-1}.\overline{\frac{n}{m}-2}}{1.\ 2.\ 3}x^3 + \frac{\frac{n}{m}.\overline{\frac{n}{m}-1}.\overline{\frac{n}{m}-2}.\overline{\frac{n}{m}-3}}{1.\ 2.\ 3.\ 4}x^4, \&c.$$

Theſe Things being premiſed, partly by way of Inſtruction, and partly by way of Exerciſe, to the young Computiſt, let us now proceed to our principal Buſineſs.

SECTION IV.

The preceding Principles applied to the Conſtruction of Logarithms.

59. Having determined § 11. that Z ſhall be greater than Unity, let $Z^1 = \overline{1+d}$, and $Z^k = \overline{1+q}$. But becauſe $Z^1 = \overline{1+d}$, therefore is $Z^k = \overline{1+d}^k = \overline{1+q}$. In order to compare theſe two values of Z^k, let each of them be expanded into a Series, the former by the Binomial Theorem § 58. and the latter as in § 14. and we ſhall have $1 + \frac{k}{1}d + \frac{k.k-1}{1.\ 2}d^2 + \frac{k.k-1.k-2}{1.\ 2.\ 3}d^3$, &c. $= 1 + n + n^2 + n^3$, &c. and $\frac{kd}{1} + \frac{k.k-1}{1.\ 2}d^2 + \frac{k.k-1.k-2}{1.\ 2.\ 3}d^3$, &c. $= n + n^2 + n^3$, &c. or $\frac{kd}{1} \times \overline{1 + \frac{k-1}{2}d + \frac{k-1.k-2}{2.\ 3}d^2}$, &c. $= n + n^2 + n^3$, &c. Say $kd \times \overline{1+x} = n + n^2 + n^3 + n^4$, &c. And then we ſhall have $kd = \frac{n + n^2 + n^3 + n^4}{1+x}$, &c. that is by § 23. $kd = \frac{n}{1} + \frac{n^2}{a} + \frac{n^3}{b} + \frac{n^4}{c} + \frac{n^5}{d} + \frac{n^6}{e}$, &c. and conſequently $2kd = \frac{2n}{1} + \frac{2n^2}{a} + \frac{2n^3}{b} + \frac{2n^4}{c} + \frac{2n^5}{d} + \frac{2n^6}{e} +$, &c.

60. In like manner ſince $Z^k = \overline{1+q}$, it follows by § 15. that $Z^{2k} = 1 + 2q + q^2$. Let each of theſe be expanded into a Series, the former by § 58. and the latter by § 15. and we ſhall have $1 + \frac{2k}{1}d + \frac{2k.2k-1}{1.\ 2}d^2 + \frac{2k.2k-1.2k-2}{1.\ 2.\ 3}d^3$, &c. $= 1 + p + p^2 + p^3$, &c. And $\frac{2k}{1}d + \frac{2k.2k-1}{1.\ 2}d^2 + \frac{2k.2k-1.2k-2}{1.\ 2.\ 3}d^3$, &c. $= p + p^2 + p^3$, &c. Or, $2kd \times \overline{1 + \frac{2k-1}{2}d + \frac{2k-1.2k-2}{2.\ 3}d^2}$, &c. $= p + p^2 + p^3 + p^4$, &c. Say $2kd \times \overline{1+y} = p + p^2 + p^3$, &c. and then we ſhall have $2kd = \frac{p + p^2 + p^3}{1+y}$, &c. which by § 23. $= \frac{p}{1} + \frac{p^2}{r} + \frac{p^3}{s} + \frac{p^4}{t} + \frac{p^5}{u} + \frac{p^6}{v}$, &c.

61. Theſe two values of $2kd$, found in § 59. and 60. muſt needs be equal; *i. e.* $\frac{2n}{1}+\frac{2n^2}{a}+\frac{2n^3}{b}+\frac{2n^4}{c}+\frac{2n^5}{d}+\frac{2n^6}{e}$, &c. $=\frac{p}{1}+\frac{p^2}{r}+\frac{p^3}{s}+\frac{p^4}{t}+\frac{p^5}{u}+\frac{p^6}{v}$, &c. But in order to compare them together, p and its Powers muſt be reduced to their original values; that is by § 15, to $\overline{2n-n^2}$, and the Powers thereof: And thus we ſhall have

$$\left.\begin{array}{l}\frac{2n}{1}\\ \frac{2n^2}{a}\\ \frac{2n^3}{b}\\ \frac{2n^4}{c}\\ \frac{2n^5}{d}\\ \frac{2n^6}{e}\\ \frac{2n^7}{f}\\ \frac{2n^8}{g}\\ \&c.\end{array}\right\}=\begin{array}{ll}\frac{2n}{1}-\frac{n^2}{1} & =\frac{p}{1}\\ +\frac{4n^2}{r}-\frac{4n^3}{r}+\frac{n^4}{r} & =\frac{p^2}{r}\\ +\frac{8n^3}{s}-\frac{12n^4}{s}+\frac{6n^5}{s}-\frac{n^6}{s} & =\frac{p^3}{s}\\ +\frac{16n^4}{t}-\frac{32n^5}{t}+\frac{24n^6}{t}-\frac{8n^7}{t}+\frac{n^8}{t} & =\frac{p^4}{t}\\ +\frac{32n^5}{u}-\frac{80n^6}{u}+\frac{80n^7}{u}-\frac{40n^8}{u},\ \&c. & =\frac{p^5}{u}\\ +\frac{64n^6}{v}-\frac{192n^7}{v}+\frac{240n^8}{v},\ \&c. & =\frac{p^6}{v}\\ +\frac{128n^7}{w}-\frac{448n^8}{w},\ \&c. & =\frac{p^7}{w}\\ +\frac{256n^8}{\omega},\ \&c. & =\frac{p^8}{\omega}\\ & \&c.\end{array}$$

62. In this Equation we find, according to § 50.

$\frac{2n}{1}=\frac{2n}{1}$: ſo that 1 is the Coefficient of n,

$\frac{2n^2}{a}=\frac{4n^2}{r}-\frac{n^2}{1}$; and $\frac{4n^2}{r}-\frac{2n^2}{a}=\frac{n^2}{1}$; Or $\frac{4}{r}-\frac{2}{a}=1$

$\frac{2n^3}{b}=\frac{8n^3}{s}-\frac{4n^3}{r}$; and $\frac{8n^3}{s}-\frac{2n^3}{b}=\frac{4n^3}{r}$; Or $\frac{8}{s}-\frac{2}{b}=\frac{4}{r}$

$\frac{2n^4}{c}=\frac{16n^4}{t}-\frac{12n^4}{s}+\frac{n^4}{r}$; and $\frac{16n^4}{t}-\frac{2n^4}{c}=\frac{12n^4}{s}-\frac{n^4}{r}$; Or $\frac{16}{t}-\frac{2}{c}=\frac{12}{s}-\frac{1}{r}$

$\frac{2n^5}{d}=\frac{32n^5}{u}-\frac{32n^5}{t}+\frac{6n^5}{s}$; and $\frac{32n^5}{u}-\frac{2n^5}{d}=\frac{32n^5}{t}-\frac{6n^5}{s}$; Or $\frac{32}{u}-\frac{2}{d}=\frac{32}{t}-\frac{6}{s}$

$\frac{2n^6}{e}=\frac{64n^6}{v}-\frac{80n^6}{u}+\frac{24n^6}{t}-\frac{n^6}{s}$; Or $\frac{64}{v}-\frac{2}{e}=\frac{80}{u}-\frac{24}{t}+\frac{1}{s}$

$\frac{2n^7}{f}=\frac{128n^7}{w}-\frac{192n^7}{v}+\frac{80n^7}{u}-\frac{8n^7}{t}$; Or $\frac{128}{w}-\frac{2}{f}=\frac{192}{v}-\frac{80}{u}+\frac{8}{t}$

$\frac{2n^8}{g}=\frac{256n^8}{\omega}-\frac{448n^8}{w}+\frac{240n^8}{v}-\frac{40n^8}{u}+\frac{n^8}{t}$; Or $\frac{256}{\omega}-\frac{2}{g}=\frac{448}{w}-\frac{240}{v}+\frac{40}{u}-\frac{1}{t}$

&c. &c.

63. Hence again, by § 52. ſince we have found that

$\frac{4}{r}-\frac{2}{a}=1$; therefore $\frac{4-2}{1}=a=r=2$

$\frac{8}{s}-\frac{2}{b}=\frac{4}{r}$; Or $\frac{8}{s}-\frac{2}{b}=2$; therefore $\frac{8-2}{2}=b=s=3$

$\frac{16}{t}-\frac{2}{c}=\frac{12}{s}-\frac{1}{r}=\frac{12}{3}-\frac{1}{2}=3\frac{1}{2}$; therefore $\frac{16-2}{3\frac{1}{2}}=\frac{32-4}{7}=c=t=4$

$\frac{32}{u}-\frac{2}{d}=\frac{32}{t}-\frac{6}{s}=\frac{32}{4}-\frac{6}{3}=6$; therefore $\frac{32-2}{6}=d=u=5$

$\frac{64}{v}-\frac{2}{e}=\frac{80}{u}-\frac{24}{t}+\frac{1}{s}=\frac{80}{5}-\frac{24}{4}+\frac{1}{3}=10\frac{1}{3}$; therefore $\frac{64-2}{10\frac{1}{3}}=e=v=6$

$\frac{128}{w}-\frac{2}{f}=\frac{192}{v}-\frac{80}{u}+\frac{8}{t}=\frac{192}{6}-\frac{80}{5}+\frac{8}{4}=18$; therefore $\frac{128-2}{18}=f=w=7$

$\frac{256}{\omega}-\frac{2}{g}=\frac{448}{w}-\frac{240}{v}+\frac{40}{u}-\frac{1}{t}=31\frac{3}{4}$; therefore $\frac{256-2}{31\frac{3}{4}}=g=\omega=8$

&c. &c.

64. Theſe few Steps are ſufficient to eſtabliſh the Law of Continuation, which gives us by § 59 and 63.

$$kd=\frac{n}{1}+\frac{n^2}{2}+\frac{n^3}{3}+\frac{n^4}{4}+\frac{n^5}{5}+\frac{n^6}{6}, \text{\&c. or } k=\frac{n}{1}+\frac{n^2}{2}+\frac{n^3}{3}+\frac{n^4}{4}+\frac{n^5}{5}+\frac{n^6}{6}, \text{\&c.}\times\frac{1}{d}: \text{\& ſo}$$

$$Z^{k}=Z^{\frac{n}{1}+\frac{n^2}{2}+\frac{n^3}{3}+\frac{n^4}{4}+\frac{n^5}{5}+\frac{n^6}{6}+\frac{n^7}{7}+\frac{n^8}{8}, \text{\&c.}\times\frac{1}{d}}=\overline{1+q}=\frac{1}{1-n}.$$

65. In the ſame manner is any other Power of Z, anſwering to a given Number, to be found. Suppoſe, for Example, $Z^{l}=\overline{1+n}$, which in § 16. was put $=\frac{1}{1-m}=1+m+m^2+m^3$, &c. And ſo

$Z^{l}=Z^{\frac{m}{1}+\frac{m^2}{2}+\frac{m^3}{3}+\frac{m^4}{4}+\frac{m^5}{5}+\frac{m^6}{6}, \text{\&c.}\times\frac{1}{d}}$. Now ſince

$Z^{k}=\frac{1}{1-n}=Z^{\frac{n}{1}+\frac{n^2}{2}+\frac{n^3}{3}+\frac{n^4}{4}+\frac{n^5}{5}+\frac{n^6}{6}, \text{\&c.}\times\frac{1}{d}}$. And

$Z^{l}=1+n=Z^{\frac{m}{1}+\frac{m^2}{2}+\frac{m^3}{3}+\frac{m^4}{4}+\frac{m^5}{5}+\frac{m^6}{6}, \text{\&c.}\times\frac{1}{d}}$. Therefore

$$Z^{l+k}=\frac{1+n}{1-n}=Z^{\frac{m+n}{1}+\frac{m^2+n^2}{2}+\frac{m^3+n^3}{3}+\frac{m^4+n^4}{4}+\frac{m^5+n^5}{5}+\frac{m^6+n^6}{6}, \text{\&c.}\times\frac{1}{d}}.$$

But by § 16.

$$m=\frac{n}{1+n}=n-n^2+n^3-n^4+n^5-n^6+n^7-n^8, \&c.$$

$$\frac{m^2}{2}=\frac{n^2}{\overline{1+n}^{2}}\times\frac{1}{2}=+\frac{n^2}{2}-\frac{2n^3}{2}+\frac{3n^4}{2}-\frac{4n^5}{2}+\frac{5n^6}{2}-\frac{6n^7}{2}+\frac{7n^8}{2}, \&c.$$

$$\frac{m^3}{3}=\frac{n^3}{\overline{1+n}^{3}}\times\frac{1}{3}=+\frac{n^3}{3}-\frac{3n^4}{3}+\frac{6n^5}{3}-\frac{10n^6}{3}+\frac{15n^7}{3}-\frac{21n^8}{3}, \&c.$$

$$\frac{m^4}{4}=\frac{n^4}{\overline{1+n}^{4}}\times\frac{1}{4}=+\frac{n^4}{4}-\frac{4n^5}{4}+\frac{10n^6}{4}-\frac{20n^7}{4}+\frac{35n^8}{4}, \&c.$$

$$\frac{m^5}{5}=\frac{n^5}{\overline{1+n}^{5}}\times\frac{1}{5}=+\frac{n^5}{5}-\frac{5n^6}{5}+\frac{15n^7}{5}-\frac{35n^8}{5}, \&c.$$

$$\frac{m^6}{6}=\frac{n^6}{\overline{1+n}^{6}}\times\frac{1}{6}=+\frac{n^6}{6}-\frac{6n^7}{6}+\frac{21n^8}{6}, \&c.$$

$$\frac{m^7}{7}=\frac{n^7}{\overline{1+n}^{7}}\times\frac{1}{7}=+\frac{n^7}{7}-\frac{7n^8}{7}, \&c.$$

$$\frac{m^8}{8}=\frac{n^8}{\overline{1+n}^{8}}\times\frac{1}{8}=+\frac{n^8}{8}, \&c.$$

So that $Z^{l}=\overline{1+n}=Z^{\frac{n}{1}-\frac{n^2}{2}+\frac{n^3}{3}-\frac{n^4}{4}+\frac{n^5}{5}-\frac{n^6}{6}+\frac{n^7}{7}-\frac{n^8}{8}, \&c.\times\frac{1}{d}}$.

But $Z^{k}=\frac{1}{1-n}=Z^{\frac{n}{1}+\frac{n^2}{2}+\frac{n^3}{3}+\frac{n^4}{4}+\frac{n^5}{5}+\frac{n^6}{6}+\frac{n^7}{7}+\frac{n^8}{8}, \&c.\times\frac{1}{d}}$. Therefore

$Z^{\overline{l+k}}=\frac{1+n}{1-n}=Z^{\frac{2n}{1}+\frac{2n^3}{3}+\frac{2n^5}{5}+\frac{2n^7}{7}, \&c.\times\frac{1}{d}}$: Or

$Z^{\overline{l+k}}=\frac{1+n}{1-n}=Z^{\frac{n}{1}+\frac{n^3}{3}+\frac{n^5}{5}+\frac{n^7}{7}, \&c.\times\frac{2}{d}}$.

66. All the ſame conſequences will follow, if we compare the Powers of Z anſwering to any two Prime Numbers, with the Power of Z anſwering to the compoſed Number, produced by multiplying thoſe two Primes together. For it is proved above, that $Z^{l}=1+n=Z^{\frac{m}{1}+\frac{m^2}{\alpha}+\frac{m^3}{\beta}+\frac{m^4}{\gamma}+\frac{m^5}{\delta}+\frac{m^6}{\varepsilon}+\frac{m^7}{\varphi}+\frac{m^8}{\chi}, \&c.\times\frac{1}{d}}$. And that $Z^{k}=\frac{1}{1-n}=Z^{\frac{n}{1}+\frac{n^2}{a}+\frac{n^3}{b}+\frac{n^4}{c}+\frac{n^5}{d}+\frac{n^6}{e}+\frac{n^7}{f}+\frac{n^8}{g}, \&c.\times\frac{1}{d}}$; and conſequently that $Z^{l+k}=\frac{1+n}{1-n}=Z^{\overline{\frac{m+n}{1}}+\overline{\frac{m^2}{\alpha}+\frac{n^2}{a}}+\overline{\frac{m^3}{\beta}+\frac{n^3}{b}}+\overline{\frac{m^4}{\gamma}+\frac{n^4}{c}}+\overline{\frac{m^5}{\delta}+\frac{n^5}{d}}+\overline{\frac{m^6}{\varepsilon}+\frac{n^6}{e}}, \&c.\times\frac{1}{d}}$. But ſince $Z^{l+k}=\overline{1+n}\times\frac{1}{1-n}=\frac{1}{1-m}\times\frac{1}{1-n}=\frac{1}{1-m-n+mn}=\frac{1}{1-R}$; therefore, by what hath been ſhewn, will $Z^{l+k}=Z^{\frac{R}{1}+\frac{R^2}{A}+\frac{R^3}{B}+\frac{R^4}{C}+\frac{R^5}{D}+\frac{R^6}{E}+\frac{R^7}{F}+\frac{R^8}{G}, \&c.\times\frac{1}{d}}$; and by comparing theſe

two values of Z^{l+k}, having firſt, for R and its Powers, put $m+n-mn$ and Powers thereof, as in § 49. we ſhall find

$$\left.\begin{matrix}
m+n \\
\frac{m^2}{\alpha}+\frac{n^2}{a} \\
\frac{m^3}{\beta}+\frac{n^3}{b} \\
\frac{m^4}{\gamma}+\frac{n^4}{c} \\
\frac{m^5}{\delta}+\frac{n^5}{d} \\
\frac{m^6}{\varepsilon}+\frac{n^6}{e} \\
\frac{m^7}{\varphi}+\frac{n^7}{f} \\
\frac{m^8}{\chi}+\frac{n^8}{g}
\end{matrix}\right\}=\left\{\begin{matrix}
R=m+n-P \\
\frac{R^2}{A}=\frac{m^2+n^2}{A}+\frac{2P}{A}-\frac{2SP}{A}+\frac{P^2}{A} \\
\frac{R^3}{B}=\frac{m^3+n^3}{B}+\frac{3SP}{B}-\frac{3S^2P}{B}+\frac{3SP^2}{B}-\frac{P^3}{B} \\
\frac{R^4}{C}=\frac{m^4+n^4}{C}\left\{\begin{matrix}+\frac{4S^2P}{C}\\-\frac{2P^2}{C}\end{matrix}\right\}-\frac{4S^3P}{C}+\frac{6S^2P^2}{C}-\frac{4SP^3}{C}+\frac{P^4}{C} \\
\frac{R^5}{D}=\frac{m^5+n^5}{D}\left\{\begin{matrix}+\frac{5S^3P}{D}\\-\frac{5SP^2}{D}\end{matrix}\right\}-\frac{5S^4P}{D}+\frac{10S^3P^2}{D}-\frac{10S^2P^3}{D},\ \&c. \\
\frac{R^6}{E}=\frac{m^6+n^6}{E}\left\{\begin{matrix}+\frac{6S^4P}{E}\\-\frac{9S^2P^2}{E}\\+\frac{2P^3}{E}\end{matrix}\right\}-\frac{6S^5P}{E}+\frac{15S^4P^2}{E},\ \&c. \\
\frac{R^7}{F}=\frac{m^7+n^7}{F}\left\{\begin{matrix}+\frac{7S^5P}{F}\\-\frac{14S^3P^2}{F}\\+\frac{7SP^3}{F}\end{matrix}\right\}-7S^6P,\ \&c. \\
\frac{R^8}{G}=\frac{m^8+n^8}{G}\left\{\begin{matrix}+\frac{8S^6P}{G}\\-\frac{20S^4P^2}{G}\\+\frac{16S^2P^3}{G}\\-\frac{2P^4}{G}\end{matrix}\right\},\ \&c.
\end{matrix}\right.$$

&c. &c. &c.

Now, ſince by § 51. the Coefficients of the ſame Powers of the ſame conſtant Quantities are equal, we have, in the firſt place, $A = \alpha = a$; $B = \beta = b$; $C = \gamma = c$; and ſo on: And, in the next place, if the equal Powers of *m* and *n* be expunged on both ſides, we ſhall have

$$\frac{2P}{A} - P = 0 \text{ ; \& thence } 2 = A = \alpha = a$$

$$\frac{3SP}{B} - \frac{2SP}{A} = 0 \text{ ; \& thence } 3 = B = \beta = b$$

$$\left.\begin{array}{l} \frac{P^2}{A} - \frac{2P^2}{C} = 0 \\ \frac{4S^2P}{C} - \frac{3S^2P}{B} = 0 \end{array}\right\} \text{; \& thence } 4 = C = \gamma = c$$

$$\left.\begin{array}{l} \frac{3SP^2}{B} - \frac{5SP^2}{D} = 0 \\ \frac{5S^3P}{D} - \frac{4S^3P}{C} = 0 \end{array}\right\} \text{; \& thence } 5 = D = \delta = d$$

$$\left.\begin{array}{l} \frac{2P^3}{E} - \frac{P^3}{B} = 0 \\ \frac{6S^2P^2}{C} - \frac{9S^2P^2}{E} = 0 \\ \frac{6S^4P}{E} - \frac{5S^4P}{D} = 0 \end{array}\right\} \text{; \& thence } 6 = E = \varepsilon = e$$

And

$$\left.\begin{array}{l} \frac{7SP^3}{F} - \frac{4SP^3}{C} = 0 \\ \frac{10S^3P^2}{D} - \frac{14S^3P^2}{F} = 0 \\ \frac{7S^5P}{F} - \frac{6S^5P}{E} = 0 \end{array}\right\} \text{; \& ſo } 7 = F = \varphi = f$$

$$\left.\begin{array}{l} \frac{P^4}{C} - \frac{2P^4}{G} = 0 \\ \frac{16S^2P^3}{G} - \frac{10S^2P^3}{D} = 0 \\ \frac{15S^4P^2}{E} - \frac{20S^4P^2}{G} = 0 \\ \frac{8S^6P}{G} - \frac{7S^6P}{F} = 0 \end{array}\right\} \text{; \& ſo } 8 = G = \chi = g$$

&c. &c.

67. Whence this way alſo it is proved, that if you put $Z^{l} = \overline{1+n} = \frac{1}{1-m}$, and $Z^{k} = \overline{1+q} = \frac{1}{1-n}$, and ſo $Z^{\overline{l+k}} = \frac{1+n}{1-n}$, then is

$$Z^{l} = \overline{1+n} = Z^{\frac{m}{1} + \frac{m^2}{2} + \frac{m^3}{3} + \frac{m^4}{4} + \frac{m^5}{5} + \frac{m^6}{6}, \text{ \&c. } \times \frac{1}{d}}\text{; which by § 65.}$$

$$\text{is equal to } Z^{\frac{n}{1} - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} + \frac{n^5}{5} - \frac{n^6}{6}, \text{ \&c. } \times \frac{1}{d}}\text{. But by § 64.}$$

$$Z^{k} = \frac{1}{1-n} = Z^{\frac{n}{1} + \frac{n^2}{2} + \frac{n^3}{3} + \frac{n^4}{4} + \frac{n^5}{5} + \frac{n^6}{6}, \text{ \&c. } \times \frac{1}{d}}\text{: Therefore is}$$

$$Z^{l+k} = \frac{1+n}{1-n} = Z^{\frac{2n}{1} + \frac{2n^3}{3} + \frac{2n^5}{5}, \text{ \&c. } \times \frac{1}{d}}\text{: Or}$$

$$Z^{l+k} = \frac{1+n}{1-n} = Z^{\frac{n}{1} + \frac{n^3}{3} + \frac{n^5}{5} + \frac{n^7}{7} + \frac{n^9}{9}, \text{ \&c. } \times \frac{2}{d}}\text{; as before in § 65.}$$

68. The improper Fraction $\frac{1+n}{1-n}$ may repreſent any improper Fraction whatever, as $\frac{r}{s}$; in which caſe, ſince $\frac{1+n}{1-n}=\frac{r}{s}$, we have $s+sn=r-rn$; and $rn+sn=r-s$; and therefore $n=\frac{r-s}{r+s}$. And thus we have a commodious Rule for finding the Power of Z anſwering to any Number whatever, directly, and independently of any other Number: For every Integer may be conſidered as an improper Fraction, whoſe Denominator is 1; and every proper Fraction as the Reciprocal of an improper one. Thus for Example, if you deſire to find the Power of Z anſwering to $\frac{2}{3}$, find Z^a anſwering to $\frac{3}{2}$; and then you have $\frac{1}{\frac{3}{2}}=\frac{Z^0}{Z^a}=Z^{-a}=\frac{2}{3}$: and ſo of others.

69. But though the Rule itſelf be general and certain, yet the operations required in the Practice of it, eſpecially in working for the firſt Primes, are ſufficiently troubleſome and tedious. However, by the Uſe of certain ſlight Artifices they may be conſiderably facilitated and ſhortened.

SECTION V.

Of the eaſieſt Ways of computing the Logarithms; with an Explanation of Dr. Halley's *facile and compendious Method.*

70. Firſt then, as $n = \frac{r-s}{r+s}$ is always a proper Fraction, it is plain that when its Numerator $\overline{r-s}$ is Unity, the Series will be raiſed with the moſt Eaſe; for then if the Denominator $r+s$ be called y, we ſhall have $n = \frac{1}{y}$, and a continual Diviſion by y^2, will ſucceſſively give all the other odd Powers of $\frac{1}{y}$ or n; each of which being afterwards divided by its proper Index, and the Quotients collected into one Sum, that Sum multiplied into $\frac{2}{a}$ willl be the required Index of the Power of Z anſwering to $\frac{r}{s}$.

71. Secondly, as the Series will be raiſed with the moſt eaſe, when $\overline{r-s}$ is Unity, ſo, the greater $\overline{r+s=y}$ is, the ſooner will it converge: that is, the fewer Steps will it take to run out at any aſſigned decimal Place.

72. Thirdly, to have $r-s=1$, $\frac{r}{s}$ muſt needs be one of the Fractions $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{5}$, *&c.* which poſſeſs this further advantage, that having found the correſponding Powers of Z, you may by multiplying thoſe Powers (which is performed, with very little trouble, by the Addition of their Indices only) find the Powers of Z anſwering to 2, 3, 4, 5; *&c.* for $\frac{3}{2} \times \frac{4}{3} = \frac{4}{2} = 2$; and $2 \times \frac{3}{2} = 3$; and $3 \times \frac{4}{3} = 4$; and $4 \times \frac{5}{4} = 5$, and $5 \times \frac{6}{5} = 6$, *&c.* But then $\overline{r+s=y}$ is ſo ſmall, towards the beginning of the Scale, that the Series converge but ſlowly, which renders the Operation ſomewhat tedious.

73. Fourthly, in order to ſecure all the abovementioned advantages at once, let it be obſerved, that if three Numbers be in Arithmetical

Progreſſion, having Unity for their common Difference, the Square of the middle Term always exceeds the Product of the Extremes by Unity. For if the Numbers be $N-1.\ N.\ N+1$, it is plain that the Square of the middle Term being N^2, and the Product of the Extremes being $N-1\times N+1=N^2-1$, the former exceeds the latter by Unity. And if $\frac{r}{s}$ be $\frac{N^2}{N^2-1}$, we ſhall have $\overline{r-s}=1$, and $\overline{r+s}=2N^2-1$.

74. To apply this to our preſent Purpoſe let $3=N$, and you will have $\frac{9}{8}=\frac{N^2}{N^2-1}$: let $4=N$, and you will have $\frac{16}{15}=\frac{N^2}{N^2-1}$: let $5=N$, and you have $\frac{25}{24}=\frac{N^2}{N^2-1}$, *&c.* Now by putting theſe new Fractions ſucceſſively for $\frac{r}{s}$, you will have firſt $\overline{r-s}=1$; ſecondly, $\overline{r+s}$ ſufficiently large to converge pretty faſt; yet, thirdly, no other Numbers are employed but thoſe which conſtitute the Fractions $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$; ſo that having found the Powers of Z correſponding to $\frac{9}{8}$, $\frac{16}{15}$, $\frac{25}{24}$, you may by multiplying thoſe Powers, that is, by adding their Indices, eaſily find the Powers of Z anſwering to the three firſt Primes, 2, 3, 5. For

$\frac{9}{8}\times\frac{16}{15}\times\frac{25}{24}=\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{5}{4}\cdot\frac{5}{6}=\frac{3}{2}\times\frac{5}{6}$ (the intermediate Terms deſtroying each other) $=\frac{3}{6}\cdot\frac{5}{2}\cdot=\frac{1}{2}\cdot\frac{5}{2}$ — — — — $=\frac{5}{4}$
Again $\frac{5}{4}\cdot\frac{16}{15}=\frac{5}{4}\cdot\frac{4}{5}\cdot\frac{4}{3}$ — — — — $=\frac{4}{3}$
And $\frac{4}{3}\cdot\frac{9}{8}=\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{3}{2}$ — — — — $=\frac{3}{2}$

Having by this indirect Method, which is by far the ſhorteſt and eaſieſt, obtained the Powers of Z correſponding to the Fractions $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, you may proceed with them, as was ſhewn in § 72. where it was propoſed to inveſtigate them directly.

75. But the Method here recommended will probably be beſt illuſtrated y an Example. Therefore let $\frac{9}{8}=\frac{r}{s}$: then is $n=\frac{r-s}{r+s}=\frac{1}{17}=\frac{1}{y}$ nd $y^2=289$.

$$\frac{1}{y}=0.05882.35294.11764.70588.235\times 1=0.05882.35294.11764\ 70588.235$$

$$\times\frac{1}{y^3}=0.00020.35416.24262.16161.204\times\frac{1}{3}=0.\text{——}6.78472.08087.38720.401$$

$$\times\frac{1}{y^5}=0.\text{——}.07042.96277.72374.260.\times\frac{1}{5}=0.\text{——}.01408.59255.54474.852$$

$$\times\frac{1}{y^7}=0.\text{——}.00024.37011.34160.464\times\frac{1}{7}=0.\text{——}.\text{——}3.48144.47737.209$$

$$\times\frac{1}{y^9}=0.\text{————}.08432.56519.586\times\frac{1}{9}=0.\text{——}.\text{——}.\text{—}936.95168.843$$

$$\times\frac{1}{y^{11}}=0.\text{——}.\text{——}.00029.17842.628\times\frac{1}{11}=0.\text{——}.\text{——}.\text{——}2.65258.420$$

$$\times\frac{1}{y^{13}}=0.\text{——}.\text{——}.\text{——}.10096.341\times\frac{1}{13}=0.\text{——}.\text{——}.\text{——}.\text{—}776.641$$

$$\times\frac{1}{y^{15}}=0.\text{——}.\text{——}.\text{——}.00034.935\times\frac{1}{15}=0.\text{——}.\text{——}.\text{——}.\text{——}2.329$$

$$\times\frac{1}{y^{17}}=0.\text{——}.\text{——}.\text{——}.\text{——}.120\times\frac{1}{17}=0.\text{——}.\text{——}.\text{——}.\text{——}.\text{—}7$$

So that $\frac{9}{8}=Z^{0.05889.15178.28191.72726.937,\ \&c.\times\frac{2}{d}}$

Now let $\frac{16}{15}=\frac{r}{s}$: then is $n=\frac{r-s}{r+s}=\frac{1}{31}=\frac{1}{y}$; and $y^2=961$.

$$\frac{1}{y}=0.03225.80645.16129.03225.806=0.03225.80645.16129.03225.806$$

$$\times\frac{1}{y^3}=0.\text{——}3.35671.84720.21751.535\times\frac{1}{3}=0.\text{——}1.11890.61573.40583.845$$

$$\times\frac{1}{y^5}=0.\text{——}.\text{—}349.29432.59127.736\times\frac{1}{5}=0.\text{——}.\text{—}69.85886.51825\ 547$$

$$\times\frac{1}{y^7}=0.\text{——}.\text{——}.36346.96419.487\times\frac{1}{7}=0.\text{——}.\text{——}.05192.42345.641$$

$$\times\frac{1}{y^9}=0.\text{——}.\text{——}.\text{—}37.82202.309\times\frac{1}{9}=0.\text{——}.\text{——}.\text{——}4.20244.701$$

$$\times\frac{1}{y^{11}}=0.\text{——}.\text{——}.\text{——}.03935.694\times\frac{1}{11}=0.\text{——}.\text{——}.\text{——}.\text{—}357.790$$

$$\times\frac{1}{y^{13}}=0.\text{——}.\text{——}.\text{——}.\text{——}4.095\times\frac{1}{13}=0.\text{——}.\text{——}.\text{——}.\text{——}.315$$

So that $\frac{16}{15}=Z^{0.03226\ 92605.68785.58583.645,\ \&c.\times\frac{2}{d}}$

Again let $\frac{25}{24} = \frac{r}{s}$: then is $n = \frac{r-s}{r+s} = \frac{1}{49} = \frac{1}{y}$; and $y^2 = 2401$.

$$\frac{1}{y} = 0.02040.81632.65306.12244.898 \qquad = 0.02040.81632.65306.12244.898$$

$$\frac{1}{3}\times\frac{1}{y^3} = 0.\text{——}.84998.59752.31408.681 \times \frac{1}{3} = 0.\text{——}.28332.86584.10469.560$$

$$\frac{1}{5}\times\frac{1}{y^5} = 0.\text{——}.\text{——}35.40133.17464.143 \times \frac{1}{5} = 0.\text{——}.\text{——}7.08026.63492.828$$

$$\frac{1}{7}\times\frac{1}{y^7} = 0.\text{——}.\text{——}.01474.44113.895 \times \frac{1}{7} = 0.\text{——}.\text{——}.\text{—}210.63444.842$$

$$\frac{1}{9}\times\frac{1}{y^9} = 0.\text{——}.\text{——}.\text{——}.61409.460 \times \frac{1}{9} = 0.\text{——}.\text{——}.\text{——}.06823.273$$

$$\frac{1}{11}\times\frac{1}{y^{11}} = 0.\text{——}.\text{——}.\text{——}.\text{——}25.576 \times \frac{1}{11} = 0.\text{——}.\text{——}.\text{——}.\text{——}2.325$$

So that $\frac{25}{24} = Z^{0.02041.09972.60127.56477.726,\&c.}$

76. Having thus found $\frac{9}{8} = Z^{0.05889.15178.28191.72726.937}$

And $\frac{16}{15} = Z^{0.03226.92605.68785.58583.645}$

And $\frac{25}{24} = Z^{0.02041.09972.60127.56477.726}$

Therefore is $\frac{9}{8}\times\frac{16}{15}\times\frac{25}{24} = \frac{5}{4} = Z^{0.11157.17756.57104.87788.308}$

And $\frac{5}{4}\times\frac{16}{15} = \frac{4}{3} = Z^{0.14384.10362.25890.46371.953}$

And $\frac{4}{3}\times\frac{9}{8} = \frac{3}{2} = Z^{0.20273.25540.54082.19098.890}$

And $\frac{3}{2}\times\frac{4}{3} = 2 = Z^{0.34657.35902.79972.65470.843}$

And $2\times\frac{3}{2} = 3 = Z^{0.54930.61443.34054.84569.733}$

And $2\times 2 = 4 = Z^{0.69314.71805.59945.30941.686}$

And $4\times\frac{5}{4} = 5 = Z^{0.80471.89562.17050.18729.994}$

And $2\times 3 = 6 = Z^{0.89587.97346.14027.50040.576}$

And $2\times 4 = 8 = Z^{1.03972.07708.39917.96412.529}$

And $3\times 3 = 9 = Z^{1.64791.84330.02164.53709.199}$

And $2\times 5 = 10 = Z^{1.15129.25464.97022.84200.837}$

&c. &c. &c.

77. By the help of theſe Powers of Z anſwering to the three firſt Prim
2, 3, 5, may the Powers of Z anſwering to the ſubſequent Primes

eaſi

ſily found. For Example, let the next Prime $7 = N$, then will $\frac{N^2}{{}^2-1} = \frac{49}{48} = \frac{r}{s}$, and $n = \frac{r-s}{r+s} = \frac{1}{97} = \frac{1}{y}$; and $y^2 = 9409$. Therefore.

$$\begin{array}{lll}
\frac{1}{y} = 0.01030.92783.50515.46391.753 & & = 0.01030.92783.50515.46391.753 \\
\times \frac{1}{y^3} = 0.\text{———}.10956.82681.52996.747 & \times \frac{1}{3} & = 0.\text{———}.03652.27560.50998.916 \\
\times \frac{1}{y^5} = 0.\text{———}.\text{———}1.16450.49224.466 & \times \frac{1}{5} & = 0.\text{———}.\text{———}.23290.09844.893 \\
\times \frac{1}{y^7} = 0.\text{———}.\text{———}.\text{———}12.37650.040 & \times \frac{1}{7} & = 0.\text{———}.\text{———}.\text{———}1.76807.148 \\
\times \frac{1}{y^9} = 0.\text{———}.\text{———}.\text{———}.\text{———}131.538 & \times \frac{1}{9} & = 0.\text{———}.\text{———}.\text{———}.\text{———}14.615 \\
\times \frac{1}{y^{11}} = 0.\text{———}.\text{———}.\text{———}.\text{———}.014 & \times \frac{1}{11} & = 0.\text{———}.\text{———}.\text{———}.\text{———}.\text{—}1
\end{array}$$

$$\text{So that } \tfrac{49}{48} = Z^{0.01030.96436.01367.84057.326 \times \frac{2}{d}}$$

$$\text{But } 48 = 6 \times 8 = Z^{1.93560.05054.53945.46453.105 \times \frac{2}{d}}$$

$$\text{Therefore } \tfrac{49}{48} \times 48 = 49 = 7^2 = Z^{1.94591.01490.55313.30510.431 \times \frac{2}{d}}$$

$$\text{And } \sqrt[2]{49} = 49^{\frac{1}{2}} = 7 = Z^{0.97295.50745.27656.65255.215 \times \frac{2}{d}}$$

And in the ſame manner may you proceed to raiſe the Powers of Z anſwering to all the other Primes, in order as they riſe, by making each Prime ſuceſſively $= N$.

78. But it is time to fix the Value of Z, and thereby that of $\frac{2}{d}$, which epends on it. Z may indeed repreſent any Number you pleaſe; but the dvantages reſulting to a Sett of Tables, from its being made equal to 10, re too many, and too well known, to be neglected. Let therefore $Z = 10$. ow, having found in § 76. the Power of Z anſwering to 10 to be $Z^{1.15129.25464.97022.84200.837, \&c. \times \frac{2}{d}}$, it is plain that theſe two Powers of Z muſt be equal, each being equal to 10; that is, that $1 = 1.15129.25464.97022.84200.837 \text{ \&c. } \times \frac{2}{d}$; and conſequently that $\frac{1}{1.15129.25464.97022.84200.837 \text{ \&c.}} = \frac{2}{d} = 0.86858.89638.06503.65530.22578.37833.21016.45887.94011.60733.31322$ &c. It remains therefore that the Indices of the ſeveral

Powers of Z, found as above, be multiplied into this Value of which though it ſeem to be a tedious taſk, is nevertheleſs to be perform with great eaſe, only by making $\frac{2}{d}$ your firſt Dividend, inſtead of 1, a proceeding in all other reſpects as before.

79. For a more convenient Notation, it may not be amiſs if you $\frac{2}{d} = 2x$; and $\frac{2x}{y} = A$; $\frac{A}{y^2} = B$; $\frac{B}{y^2} = C$; $\frac{C}{y^2} = D$, &c. Thus for $\frac{9}{8}$ you ha

$$2x = 0.86858.89638.06503.65530.226;\ y = 17;\ \text{and}\ y^2 = 289.$$

$$A = 0.05109.34684.59206.09737.072 \qquad = 0.05109.34684.59206.09737.072$$

$$\frac{1}{3}B = 0.\text{——}17.67940.08509.36365.872 \times \frac{1}{3} = 0.\text{——}5.89313.36169.78788.624$$

$$\frac{1}{5}C = 0.\text{———}.06117.43974.08084.311 \times \frac{1}{5} = 0.\text{———}.01223.48794.81616.862$$

$$\frac{1}{7}D = 0.\text{———}.\text{——}21.16761.15598.907 \times \frac{1}{7} = 0.\text{———}.\text{———}3.02394.45085.558$$

$$\frac{1}{9}E = 0.\text{———}.\text{———}.07324.43306.570 \times \frac{1}{9} = 0.\text{———}.\text{———}.\text{—}813.82589.619$$

$$\frac{1}{11}F = 0.\text{———}.\text{———}.\text{——}25.34405.905 \times \frac{1}{11} = 0.\text{———}.\text{———}.\text{———}2.30400.537$$

$$\frac{1}{13}G = 0.\text{———}.\text{———}.\text{———}.08769.570 \times \frac{1}{13} = 0.\text{———}.\text{———}.\text{———}.\text{—}674.582$$

$$\frac{1}{15}H = 0.\text{———}.\text{———}.\text{———}.\text{——}30.344 \times \frac{1}{15} = 0.\text{———}.\text{———}.\text{———}.\text{———}2.025$$

$$\frac{1}{17}I = 0.\text{———}.\text{———}.\text{———}.\text{———}.105 \times \frac{1}{17} = 0.\text{———}.\text{———}.\text{———}.\text{———}.\text{—}6$$

So that when $10 = Z^1$, then $\frac{9}{8} = Z^{0.05115.25224.47381.28894.88}$, &

Again for $\frac{16}{15}$ you have $y = 31$; $y^2 = 961$

$2x$ = 0.86858.89638.06503.65530.226

A = 0.02801.89988.32467.85984.846   0.02801.89988.32467.85984.846

$\frac{1}{3}$ B = 0.——2.91560.86194.03523.397 × $\frac{1}{3}$ = 0.——.97186.95398.01174.466

$\frac{1}{5}$ C = 0.——.—303.39319.66080.669 × $\frac{1}{5}$ = 0.——.—60.67863.93216.134

$\frac{1}{7}$ D = 0.——.——.31570.57196.754 × $\frac{1}{7}$ = 0.——.——.04510.08170.965

$\frac{1}{9}$ E = 0.——.——.—32.85179.185 × $\frac{1}{9}$ = 0.——.——.——3.65019.909

$\frac{1}{11}$ F = 0.——.——.——.03418.500 × $\frac{1}{11}$ = 0.——.——.——.—310.772

$\frac{1}{13}$ G = 0.——.——.——.——3.557 × $\frac{1}{13}$ = 0.——.——.——.——.273

So that when $10 = Z^1$, then $\frac{16}{15} = Z^{0.02802.87236.00243.53877.37}$, &c.

And for $\frac{25}{24}$ you have $y = 49$; $y^2 = 2401$

$2x$ = 0.86858.89638.06503.65530.226

A = 0.01772.63053.83806.19704.698   = 0.01772.63053.83806.19704.698

B = 0.——.73828.84374.76309.747 × $\frac{1}{3}$ = 0.——.24609.61458.25436.582

C = 0.——.—30.74920.60589.884 × $\frac{1}{5}$ = 0.——.——6.14984.12117.977

D = 0.——.——.01280.68330.108 × $\frac{1}{7}$ = 0.——.——.—182.95475.730

E = 0.——.——.——.53338.579 × $\frac{1}{9}$ = 0.——.——.——.05926.620

F = 0.——.——.——.—22.215 × $\frac{1}{11}$ = 0.——.——.——.——2.019

So that when $10 = Z^1$, then is $\frac{25}{24} = Z^{0.01772.87669.60431.58663.63}$, &c.

80. Now ſince $\frac{9}{8}$ —— $= \mathrm{Z}^{0.05115.25224.47381.28894.88}$

And $\frac{16}{15}$ —— $= \mathrm{Z}^{0\ 02802.87236.00243.53877.37}$

And $\frac{25}{24}$ —— $= \mathrm{Z}^{0.01772.87669.60431.58663.63}$

We have $\frac{9}{8} \times \frac{16}{15} \times \frac{25}{24} = \frac{5}{4} = \mathrm{Z}^{0.09691.00130.08056.41435.88}$

And $\frac{5}{4} \times \frac{16}{15} = \frac{4}{3} = \mathrm{Z}^{0.12493.87366.08299.95313.25}$

And $\frac{4}{3} \times \frac{9}{8} = \frac{3}{2} = \mathrm{Z}^{0.17609.12590.55681.24208.12}$

And $\frac{3}{2} \times \frac{4}{3} = 2 = \mathrm{Z}^{0.30102.99956.63981.19521.37}$

And $2 \times \frac{3}{2} = 3 = \mathrm{Z}^{0.47712.12547.19662.43729.50}$

And $3 \times \frac{4}{3} = 4 = \mathrm{Z}^{0.60205.99913.27962.39042.75}$

And $4 \times \frac{5}{4} = 5 = \mathrm{Z}^{0.69897.00043.36018.80478.63}$

And ſo $2 \times 5 = 10 = \mathrm{Z}^{1.}$

81. After what hath been ſaid there cannot, I think, be any doubt concerning the manner in which the Powers of Z, anſwering to the other Primes are to be raiſed. But Dr. Halley's facile and compendious method of performing the requiſite Operations remains yet to be explained.

By § 70. If $n = \frac{1}{y}$ then $\mathrm{Z}^{k} = \mathrm{Z}^{\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{1}{9y^9}, \&c. \times 2x}$, and the 1ſt Term in the Series for finding k being $\frac{2x}{y}$, let $\frac{2x}{y} = \mathrm{A}$

2d is $\frac{2x}{3y^3} = \frac{2x}{y} \times \frac{1}{3y^2} = \mathrm{A} \times \frac{1}{3y^2} = \frac{\mathrm{A}}{3y^2} = \mathrm{B}$

3d is $\frac{2x}{5y^5} = \frac{2x}{3y^3} \times \frac{1}{\frac{5}{3}y^2} = \mathrm{B} \times \frac{1}{\frac{5}{3}y^2} = \frac{\mathrm{B}}{\frac{5}{3}y^2} = \mathrm{C}$

4th is $\frac{2x}{7y^7} = \frac{2x}{5y^5} \times \frac{1}{\frac{7}{5}y^2} = \mathrm{C} \times \frac{1}{\frac{7}{5}y^2} = \frac{\mathrm{C}}{\frac{7}{5}y^2} = \mathrm{D}$

5th is $\frac{2x}{9y^9} = \frac{2x}{7y^7} \times \frac{1}{\frac{9}{7}y^2} = \mathrm{D} \times \frac{1}{\frac{9}{7}y^2} = \frac{\mathrm{D}}{\frac{9}{7}y^2} = \mathrm{E}$ and ſo on.

82. Now the Diviſors here employed are very eaſily found: for the Diviſor of $2x$ is y; that of A is $y^2 + \frac{2y^2}{1} = 3y^2$;

$$\begin{aligned}
&\text{of B is } y^2 + \frac{2y^2}{3} = \tfrac{5}{3}y^2;\\
&\text{of C is } y^2 + \frac{2y^2}{5} = \tfrac{7}{5}y^2;\\
&\text{of D is } y^2 + \frac{2y^2}{7} = \tfrac{9}{7}y^2;\ \&c.
\end{aligned}$$

83. For an Example of this Method, let us find the Power of Z anſwering to $\frac{9}{8}$; where $y = 17$, and $y^2 = 289$, and $2x = 0.86858.89638.06503.65530.226$.

$$\begin{array}{lllll}
\frac{2x}{y} = \frac{2x}{17} & \text{gives} & A & = & 0.05109.34684.59206.09737.072\\
\frac{A}{3y^2} = \frac{A}{867} & \text{gives} & B & = & 0.\text{———}5.89313.36169.78788.624\\
\frac{B}{\frac{5}{3}y^2} = \frac{B}{481\frac{2}{3}} & \text{gives} & C & = & 0.\text{———}.01223.48794.81616.862\\
\frac{C}{\frac{7}{5}y^2} = \frac{C}{404.6} & \text{gives} & D & = & 0.\text{———}.\text{———}3.02394.45085.558\\
\frac{D}{\frac{9}{7}y^2} = \frac{D}{371\frac{4}{7}} & \text{gives} & E & = & 0.\text{———}.\text{———}.\text{—}813.82589.619\\
\frac{E}{\frac{11}{9}y^2} = \frac{E}{353\frac{2}{9}} & \text{gives} & F & = & 0.\text{———}.\text{———}.\text{———}2.30400.537\\
\frac{F}{\frac{13}{11}y^2} = \frac{F}{341\frac{6}{11}} & \text{gives} & G & = & 0.\text{———}.\text{———}.\text{———}.\text{—}674.582\\
\frac{G}{\frac{15}{13}y^2} = \frac{G}{333\frac{6}{13}} & \text{gives} & H & = & 0.\text{———}.\text{———}.\text{———}.\text{———}2.025\\
\frac{H}{\frac{17}{15}x^2} = \frac{G}{327\frac{8}{15}} & \text{gives} & I & = & 0.\text{———}.\text{———}.\text{———}.\text{———}.\text{—}6\\
\hline
\end{array}$$

So that $\frac{9}{8}$ is, as found above § 79, $= Z^{0.05115.25224.47381.28894.88,\ \&c.}$

84. Thus have we ſufficiently eſtabliſhed the Arithmetical Principles, and explained the beſt Practical Methods thereon founded, for raiſing the Power of Z anſwering to any given Number, let Z be what it will. And now, if the Tables propoſed in § 12 be accordingly conſtructed, the Letter Z may be dropt, as an encumbrance in working, while the Indices remain, under the Title of Logarithms, to be uſed juſt in the ſame manner as if Z were preſent.

SECTION VI.

To find the Number from its Logarithm given.

85. The Converſe of the preceding Problem comes next to be conſidered; that is, in the equation $Z^{k} = \overline{1+q}$, having k given, to find $\overline{1+q}$.

The Value of $d = \frac{1}{x}$ having been determined § 78, and k being given, we have thence $kd = \frac{k}{x}$, which call L. Now ſince by § 14, if $\frac{q}{1+q} = n$, then is $q = n + n^{2} + n^{3} + n^{4}$, &c. and by § 64, kd or $L = n + \frac{n^{2}}{2} + \frac{n^{3}}{3} + \frac{n^{4}}{4}$, &c. whence it follows that L is greater than n; and alſo that $L + L^{2} + L^{3} + L^{4}$, &c. is greater than $n + n^{2} + n^{3} + n^{4}$, &c. that is, greater than q. Therefore q the leſs Number is ſome part or parts of $L + L^{2} + L^{3} + L^{4}$, the greater Number: ſuppoſe $q = \frac{L + L^{2} + L^{3} + L^{4} + L^{5} + L^{6}}{1 + \varphi}$, &c. which by § 23, may be expreſſed thus, $L + \frac{L^{2}}{x} + \frac{L^{3}}{y} + \frac{L^{4}}{z} + \frac{L^{5}}{u} + \frac{L^{6}}{w}$; then will $1 + 2q + q^{2}$, the Square of $\overline{1+q}$ be equal to the Square of $1 + L + \frac{L^{2}}{x} + \frac{L^{3}}{y} + \frac{L^{4}}{z} + \frac{L^{5}}{u} + \frac{L^{6}}{w}$, &c. that is,

$$\overline{1+2q+q}^{2} = \left\{ \begin{array}{l} 1 + \frac{2L}{1} + \frac{2L^{2}}{x} + \frac{2L^{3}}{y} + \frac{2L^{4}}{z} + \frac{2L^{5}}{u} + \frac{2L^{6}}{w} \\ \qquad\quad + \frac{L^{2}}{1} + \frac{2L^{3}}{x} + \frac{L^{4}}{x^{2}} + \frac{2L^{5}}{xy} + \frac{L^{6}}{y^{2}} \\ \qquad\qquad\qquad\qquad + \frac{2L^{4}}{y} + \frac{2L^{5}}{z} + \frac{2L^{6}}{xz} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \frac{2L^{6}}{u} \end{array} \right\} \text{\&c.}$$

86. Again, by § 15, if $1+2q+q^2=\frac{1}{1-p}$, then is $\overline{2q+q^2}=p+p^2+p^3+p^4$, &c. and $2kd$ or $2\mathrm{L}$, by § 65, $=p+\frac{p^2}{2}+\frac{p^3}{3}+\frac{p^4}{4}$, &c. So that $2\mathrm{L}$ is greater than p; and conſequently $2\mathrm{L}+4\mathrm{L}^2+8\mathrm{L}^3+16\mathrm{L}^4$, &c. is greater than $p+p^2+p^3+p^4$, &c. which is equal to $2q+q^2$. Therefore $2q+q^2$, being the leſs Number, muſt be ſome part or parts of the greater Number $2\mathrm{L}+4\mathrm{L}^2+8\mathrm{L}^3+16\mathrm{L}^4$, &c. ſuppoſe $2q+q^2=\frac{2\mathrm{L}+4\mathrm{L}^2+8\mathrm{L}^3+16\mathrm{L}^4}{1+x}$, &c. then by § 23 we may put $1+2q+q^2=1+2\mathrm{L}+\frac{4\mathrm{L}^2}{\alpha}+\frac{8\mathrm{L}^3}{\beta}+\frac{16\mathrm{L}^4}{\gamma}+\frac{32\mathrm{L}^5}{\delta}+\frac{64\mathrm{L}^6}{\epsilon}$, &c.

87. Theſe two Values of $1+2q+q^2$ being compared together give the following Equation,

$$\begin{array}{l} 1+2\mathrm{L} \\ +\frac{4\mathrm{L}^2}{\alpha} \\ +\frac{8\mathrm{L}^3}{\beta} \\ +\frac{16\mathrm{L}^4}{\gamma} \\ +\frac{32\mathrm{L}^5}{\delta} \\ +\frac{64\mathrm{L}^6}{\epsilon} \\ \text{\&c.} \end{array} = \left\{\begin{array}{l} 1+\frac{2\mathrm{L}}{1} \\ +\frac{2\mathrm{L}^2}{x}+\frac{\mathrm{L}^2}{1} \\ +\frac{2\mathrm{L}^3}{y}+\frac{2\mathrm{L}^3}{x} \\ +\frac{2\mathrm{L}^4}{z}+\frac{\mathrm{L}^4}{x^2}+\frac{2\mathrm{L}^4}{y} \\ +\frac{2\mathrm{L}^5}{u}+\frac{2\mathrm{L}^5}{xy}+\frac{2\mathrm{L}^5}{z} \\ +\frac{2\mathrm{L}^6}{w}+\frac{\mathrm{L}^6}{y^2}+\frac{2\mathrm{L}^6}{xz}+\frac{2\mathrm{L}^6}{u} \\ \text{\&c.} \end{array}\right.$$

Whence by § 50 it follows that,

$$\begin{array}{l} \frac{4}{\alpha}=\frac{2}{x}+1 \\ \frac{8}{\beta}=\frac{2}{y}+\frac{2}{x} \\ \frac{16}{\gamma}=\frac{2}{z}+\frac{1}{x^2}+\frac{2}{y} \\ \frac{32}{\delta}=\frac{2}{u}+\frac{2}{xy}+\frac{2}{z} \\ \frac{64}{\epsilon}=\frac{2}{w}+\frac{1}{y^2}+\frac{2}{xz}+\frac{2}{u} \\ \text{\&c.} \end{array}$$

88. Whence again by § 52, since $\frac{4}{\alpha}=\frac{2}{x}+1$; therefore $\frac{a-b}{c}=\frac{4-2}{1}=\alpha=x=2$

And ſince $\frac{8}{\beta}=\frac{2}{y}+\frac{2}{x}$; or $\frac{8}{\beta}=\frac{2}{y}+1$; therefore $\frac{a-b}{c}=\frac{8-2}{1}=\beta=y=6$

And ſince $\frac{16}{\gamma}=\frac{2}{z}+\frac{1}{x^2}+\frac{2}{y}$; or $\frac{16}{\gamma}=\frac{2}{z}+\frac{7}{12}$; therefore $\frac{a-b}{c}=\frac{16-2}{7}\times 12=\gamma=z=24$

And ſince $\frac{32}{\delta}=\frac{2}{u}+\frac{2}{xy}=\frac{2}{z}$; or $\frac{32}{\delta}=\frac{2}{u}+\frac{1}{4}$; therefore $\frac{a-b}{c}=\frac{32-2}{1}\times 4=\delta=u=120$

And ſince $\frac{64}{\epsilon}=\frac{2}{w}+\frac{1}{y^2}+\frac{2}{xz}+\frac{2}{u}$; or $\frac{64}{\epsilon}=\frac{2}{w}+\frac{31}{360}$ then $\frac{a-b}{c}=\frac{64-2}{31}\times 360=\epsilon=w=720$.

&c. &c.

89. Theſe few Steps fully prove that when $Z^k = \overline{1+q}$, if you put $kd = L$, then is $1+q = 1 + \frac{L}{1} + \frac{L^2}{1.2} + \frac{L^3}{1.2.3} + \frac{L^4}{1.2.3.4} + \frac{L^5}{1.2.3.4.5} + \frac{L^6}{1.2.3.4.5.6}$, &c. and ſo on, the Law of Continuation being evident.

90. Hence alſo ſince $Z^k = 1+q$, and $\frac{Z^0}{Z^k} = Z^{-k} = \frac{1}{1+q} = 1-n$, by § 14, we have $1-n = \frac{1}{1+L+\frac{L^2}{2}+\frac{L^3}{6}+\frac{L^4}{24}, \&c.} = 1 - \frac{L}{1} + \frac{L^2}{1.2} - \frac{L^3}{1.2.3} + \frac{L^4}{1.2.3.4} - \frac{L^5}{1.2.3.4.5} + \frac{L^6}{1.2.3.4.5.6}$, &c.

91. And if the Terms in which L and its Powers are ſucceſſively preſent be called A, B, C, D, &c. then will

$$1+q = 1+L+A.\frac{L}{2}+B.\frac{L}{3}+C.\frac{L}{4}+D.\frac{L}{5}+E.\frac{L}{6}, \&c.$$

$$\text{and } 1-n = 1-L-A.\frac{L}{2}-B.\frac{L}{3}-C.\frac{L}{4}-D.\frac{L}{5}-E.\frac{L}{6}, \&c.$$

92. It muſt be owned that this Series converges very ſlowly, eſpecially when L is large. In order to remedy this in ſome ſort let it be obſerved, ſince $Z^k = \overline{1+q}$, that if we ſuppoſe $Z^{k-r} = a$, or $Z^{k+r} = b$, then will $\frac{Z^k}{Z^{k-r}} = \frac{1+q}{a} = Z^{+r}$; and $\overline{1+q} = a \times Z^r$: Or $\frac{Z^k}{Z^{k+r}} = \frac{1+q}{b} = Z^{-r}$; and $\overline{1+q} = b \times Z^{-r}$. Now let rd, or $\frac{r}{x} = l$, and by § 89 $\frac{1+q}{a} = Z^r = 1 + \frac{l}{1} + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120}$, &c. and therefore $1+q = a + \frac{al}{1} + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120}$, &c. Or, ſince $\frac{1+q}{b} = Z^{-r} =$ by § 90 to $1 - \frac{l}{1} + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120}$, &c. therefore is $1+q = b - \frac{bl}{1} + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120}$, &c.

93. If therefore you are provided with good Tables of Logarithms, carried to a ſufficient number of places, (for you cannot expect to find a number that has more places than L or l), ſeek therein the Loga-

rithm neareſt to k, whether greater or leſs, and its difference from k call r. If the Logarithm you pitch upon be leſs than k call it $k-r$, and its correſpondent number call a; if it be greater than k, call it $k+r$, and its correſponding number b: then making $rd=\frac{r}{x}=l$, with a and $+l$, or b and $-l$, proceed as above, § 92.

94. If l be ſmall, this Proceſs may be conſiderably abridged, by by taking $\left.\begin{array}{l}\frac{1+q}{a}=1+\frac{l}{1-\frac{1}{2}l}-\frac{\frac{1}{12}l^3}{1-l}+\frac{\frac{1}{35}l^5}{1-2l}, \text{\&c. or} \\ \frac{1+q}{b}=1-\frac{l}{1+\frac{1}{2}l}+\frac{\frac{1}{12}l^3}{1+l}-\frac{\frac{1}{35}l^5}{1+2l}, \text{\&c.}\end{array}\right\}$ which are eaſily deduced from $1 \pm l+\frac{l^2}{2} \pm \frac{l^3}{6}+\frac{l^4}{24} \pm \frac{l^5}{120}+\frac{l^6}{720}$, &c. For, with regard to $\frac{1+q}{a}$, it is plain that $l+\frac{l^2}{2}=l\times\overline{1+\frac{1}{2}l}$. Now $\frac{1}{1+\frac{1}{2}l}=1-\frac{1}{2}l+\frac{1}{4}l^2-\frac{1}{8}l^3$, &c. and therefore $\overline{1+\frac{1}{2}l}=\frac{1}{1-\frac{1}{2}l+\frac{1}{4}l^2-\frac{1}{8}l^3}$, &c. and conſequently $\frac{l}{1-\frac{1}{2}l+\frac{1}{4}l^2-\frac{1}{8}l^3}=\overline{l+\frac{l^2}{2}}$. But the Sum of theſe two, with 1, is leſs than the Series for $\frac{1+q}{a}$ in § 92; and if to encreaſe the Quotient you leſſen the Diviſor, then $\frac{l}{1-\frac{1}{2}l}$ will give you $l+\frac{l^2}{2}+\frac{l^3}{4}+\frac{l^4}{8}+\frac{l^5}{16}+\frac{l^6}{32}$, &c. Now this is too much by $\frac{1}{12}l^3+\frac{1}{12}l^4+\frac{13}{240}l^5+\frac{43}{1440}l^6$, &c. which muſt therefore be ſubtracted. The two firſt Terms, $\frac{l^3+l^4}{12}=\frac{1}{12}l^3\times\overline{1+l}$: but $\overline{1+l}=\frac{1}{1-l+l^2-l^3, \text{\&c.}}$: therefore $\frac{l^3+l^4}{12}=\frac{\frac{1}{12}l^3}{1-l+l^2-l^3, \text{\&c.}}$, the Sum whereof being leſs than the Series to be ſubtracted, encreaſe the Quotient, as before, by leſſening the Diviſor, and ſubtract $\frac{\frac{1}{12}l^3}{1-l}=\frac{1}{12}l^3+\frac{1}{12}l^4+\frac{1}{12}l^5+\frac{1}{12}l^6$, &c. But now you have ſubtracted too much by $\frac{7}{240}l^5+\frac{77}{1440}l^6$, &c. $=\frac{7}{240}l^5\times\overline{1+\frac{11}{6}l}$, or nearly $\frac{1}{34}l^5\times\overline{1+2l}$; or $\frac{1}{34}l^5\times\frac{1}{1-2l+4l^2-8l^3, \text{\&c.}}$ $=\frac{\frac{1}{34}l^5}{1-2l+4l^2-8l^3, \text{\&c.}}=\frac{7}{240}l^5+\frac{77}{1440}l^6$. But the Sum of theſe

being leſs than the Series to be added, leſſen the Diviſor as before, and add $\frac{\frac{1}{34}l^5}{1-2l} = \frac{7}{238}l^5 + \frac{77}{1309}l^6$, &c. which being a little too much, it may be leſſened to $\frac{\frac{1}{35}l^5}{1-2l}$: and ſo you may proceed further if you pleaſe.

95. The Series for $\frac{1+q}{b}$ muſt evidently come out the very ſame, except only that all the Signs of l and its powers will be changed, as appears by the following work.

$$\frac{1+q}{b} = 1-l+\frac{l^2}{2}-\frac{l^3}{6}+\frac{l^4}{24}-\frac{l^5}{120}+\frac{l^6}{720}, \text{\&c. But}$$

$$1-\frac{l}{1+\frac{1}{2}l} \text{ or } \frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} = 1-l+\frac{l^2}{2}-\frac{l^3}{4}+\frac{l^4}{8}-\frac{l^5}{16}+\frac{l^6}{32}, \text{\&c. therefore}$$

$$\frac{1+q}{b}-\frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} = \frac{1}{12}l^3-\frac{1}{12}l^4+\frac{13}{240}l^5-\frac{43}{1440}l^6, \text{\&c. Or}$$

$$\frac{1+q}{b}-\frac{1}{12}l^3+\frac{1}{12}l^4-\frac{13}{240}l^5+\frac{43}{1440}l^6, \text{\&c.} = \frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} = 1-\frac{l}{1+\frac{1}{2}l} \text{ Add}$$

$$+\frac{1}{12}l^3-\frac{1}{12}l^4+\frac{1}{12}l^5-\frac{1}{12}l^6, \text{\&c.} = \frac{\frac{1}{12}l^3}{1+l} \text{ And the Sum is}$$

$$\frac{1+q}{b} + \frac{7}{240}l^5-\frac{77}{1440}l^6, \text{\&c.} = 1-\frac{l}{1+\frac{1}{2}l}+\frac{\frac{1}{12}l^3}{1+l}$$

$$\text{Subtract} \quad +\frac{1}{35}l^5-\frac{2}{35}l^6, \text{\&c.} = \frac{\frac{1}{35}l^5}{1+2l} \text{ And there remains}$$

$$\frac{1+q}{b} + \frac{1}{1680}l^5-\frac{37}{10080}l^6, \text{\&c.} = 1-\frac{l}{1+\frac{1}{2}l}+\frac{\frac{1}{12}l^3}{1+l}-\frac{\frac{1}{35}l^5}{1+2l}, \text{\&c.}$$

96. If l be very ſmall, and your Logarithms extend to many places, the firſt ſtep alone may ſuffice where great Accuracy is not required; and in that caſe the given Logarithm, whatever Species it be of, provided $x = \frac{1}{d}$ be known, may be employed, without reducing it previouſly to the Natural Logarithm, by an actual Diviſion thereof by x, or Multiplication thereof into d. For ſince $l = \frac{r}{x} = rd$, put that value in its ſtead, and $\frac{l}{1-\frac{1}{2}l}$ becomes $\frac{rd}{1-\frac{1}{2}rd}, = \frac{r}{\frac{1}{d}-\frac{1}{2}r} = \frac{r}{x-\frac{1}{2}r}$; and ſo $\frac{al}{1-\frac{1}{2}l} = \frac{ar}{x-\frac{1}{2}r}$; and $a + \frac{ar}{x-\frac{1}{2}r} = \frac{ax+\frac{1}{2}ar}{x-\frac{1}{2}r} = \overline{1+q} = b - \frac{br}{x+\frac{1}{2}r} = \frac{bx-\frac{1}{2}br}{x+\frac{1}{2}r}$. Therefore ſince $\frac{ax+\frac{1}{2}ar}{x-\frac{1}{2}r} = \overline{1+q}$; or $\overline{x+\frac{1}{2}r} \times a = \overline{x-\frac{1}{2}r} \times \overline{1+q}$, we have $\overline{x-\frac{1}{2}r} : \overline{x+\frac{1}{2}r} :: a : \overline{1+q}$. Alſo ſince $\frac{bx-\frac{1}{2}br}{x+\frac{1}{2}r} = 1+q$; or $\overline{x-\frac{1}{2}r} \times b = \overline{x+\frac{1}{2}r} \times \overline{1+q}$, therefore is $\overline{x+\frac{1}{2}r} : \overline{x-\frac{1}{2}r} :: b : \overline{1+q}$; all which is perfectly agreeable to the great Halley's Doctrine in the Philoſ. Tranſ. N° 216.

Thus, I hope, I have clearly made appear what I propoſed to ſhew; namely, that the Doctrine of Logarithms depends wholly on, and is moſt naturally and eaſily deducible from the plaineſt Arithmetical Principles; without calling to our aid, either the Properties of the Hyperbola, on one hand, or the Subtilties of Indiviſibles, Fluxions, Ratios, the Roots of Infinite Series, on the other. The Hyperbola cannot be applied to this Subject, till it be reduced from a Geometrical Curve to an Algebraic Equation; and the other Methods above-mentioned, though originally and purely Arithmetical, require ſomething more than an ordinary Capacity to underſtand them readily and ſatisfactorily; eſpecially as ſome of the greateſt Maſters have, in treating them, expreſſed themſelves ſo obſcurely, as to render their otherwiſe admirable works of very little uſe to young beginners; which I flatter myſelf will not be amongſt the many Objections that may be juſtly be made to this Performance.

FINIS.

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ERRATA.

Page 11. § 45. *l.* 1. for $1+\frac{x^m}{y}$ *read* $1+\frac{x^m}{y^m}$

Page 18. § 55. *l. ult.* for $\frac{{}^{\frac{n}{m}}\overline{1+x} - \overline{1+y}^{\frac{n}{m}}}{\overline{1+x} - \overline{1+y}}$ *read* $\frac{\overline{1+x}^{\frac{n}{m}} - \overline{1+y}^{\frac{n}{m}}}{\overline{1+x} - \overline{1+y}}$